Sensitivity Analysis of Maximum Entropy Model

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1. Abstract

We consider the maximum entropy problem with linear equality constraints

$$\text{ME} : \max_{Ap=c} \sum p_i \log \frac{p_i}{q_i} \tag{1.1}$$

where $A = (a_{ij}) \in \mathbb{R}^{m \times n}, q = (q_j) \in \mathbb{R}^n$ and $c = (c_i) \in \mathbb{R}^m$, with $m < n, q_i > 0$ and $c_i > 0$. Suppose p^* is the solution of (1.1), then it is easy to verify that $p^* = \pi_j \prod_{s=1}^n t_s^{a_{sj}}$ for some $t = (t_1, ..., t_n)$, where $t_s > 0$ and where $\pi_j = e^{\log q_j - 1}$. We will assume that $Ap = c$ has a position solution and $rank(A) = m$. The assumption that $Ap = c$ has a positive solution ensures the unique existence of p^* [6]. The condition that $rank(A) = m$ guarantees that there is a unique $t = (t_1, ..., t_n)$ such that $p^* = \pi_j \prod_{s=1}^n t_s^{a_{sj}}$.

The **ME** problem arises in various applications, including the multidimensional contingency table computation [4], self assembly[5] and natural language processing [1]. When the constraints that

$$\sum_{s=1}^n c_s = 1, \sum_{s=1}^n a_{sj} = 1, \text{and } a_{sj} \geq 0, \text{ for all } s, j. \tag{1.2}$$

a standard method solving (1.1) is Iterative Proportional Scaling method [3], and this method is extended in [2], where all restrictions in (1.2) are removed.

The result we present in this paper is the sensitivity analysis of the solution of (1.1). As in both Iterative Proportional Scaling method and extended Iterative Proportional Scaling method, the computed solution $\hat{p}$ satisfies $\hat{p} = \pi_j \prod_{s=1}^n t_s^{a_{sj}}$ and $A\hat{p} = \hat{c}$, therefore we are specially interested in the sensitivity analysis of p to perturbations of $c = (c_1, ..., c_n)$.

Our perturbation result for the ME problem is summarized as follows:

Proposition 1.1. *Suppose the ME problem (1.1) is solved using the extended Iterative Proportional Scaling method. Let* $\hat{p} = \pi_j \prod_{s=1}^n \hat{t_s}^{a_{sj}}$ *denote the computed*

solution and $p^* = \pi_j \prod_{s=1}^{n} {t_s^*}^{a_{sj}}$ *is the solution of* (1.1)*. Let* $A\hat{p} = \hat{c}$*,* $\Delta c = c - \hat{c}$*,* $\Delta p = p^* - \hat{p}$ *and* $\epsilon = \frac{\|\Delta c\|}{\|c\|}$*. Then*

$$\frac{\| \Delta p \|}{\| p^* \|} \leq \| A^T [A \begin{pmatrix} p_1^* & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & p_m^* \end{pmatrix} A^T]^{-1} \| \| c \| \epsilon + O(\epsilon^2).$$

From the perspective of application, the sensitivity analysis of p to perturbations of $c = (c_1, ..., c_n)$ arises in self assembly, where p^* represents the concentration of each species in the system at the equilibrium state where $\Delta c = A\Delta x$ and Δx is the initial condition perturbation for each species. The sensitivity analysis of p to perturbations of $c = (c_1, ..., c_n)$ also arises in natural language processing. Moreover, it is helpful for backward error estimation in real computation.

References

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