On the computation of A_{∞} -maps

A. Berciano, M.J. Jiménez, P. Real

Universidad de Sevilla

September 2007

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Notion of A_{∞} -structures.

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Notion of A_{∞} -structures.

Theoretical results to compute them.

A. Berciano, M.J. Jiménez, P. Real On the computation of A_{∞} -maps

< □ > < 同 > < 三 >

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э



Notion of A_{∞} -structures.

Theoretical results to compute them.

Improvements in the theoretical algorithms.

(日)

Index

Introduction

- Preliminaries
- Historical origin
- Mathematical notion

2 A_{∞} -structures via perturbation

- Contractions
- The tensor trick

3 Theoretical study of complexity

- Main Results
- Computational advantages

Preliminaries Historical origin Mathematical notion

Index

1 Introduction

- Preliminaries
- Historical origin
- Mathematical notion

2) A_∞ -structures via perturbation

- Contractions
- The tensor trick

3 Theoretical study of complexity

- Main Results
- Computational advantages

< 17 ▶

Introduction	Preliminaries
A_∞ -structures via perturbation	Historical origin
Theoretical study of complexity	Mathematical notion

Rings

Let us suppose that Λ is a commutative ring with $1 \neq 0$.

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Introduction	Preliminaries
A_∞ -structures via perturbation	Historical origin
Theoretical study of complexity	

Rings

Let us suppose that Λ is a commutative ring with $1 \neq 0$.

Modules

A *DG-module* is a graded module $M = \{M_n\}_{n\geq 0}$, endowed with a differential $d: M \to M$ (that is, a morphism of graded modules of degree -1 such that $d^2 = 0$).

Preliminaries Historical origin Mathematical notion

Algebras

Definition

A differential graded algebra (A, μ_A, η) , or simply DG-algebra, is a DG-module equipped with two morphisms $\mu_A : A \otimes A \to A$ and $\eta : \Lambda \to A$, such that μ_A is an *associative product*, i.e., $\mu_A(\mu_A \otimes 1) = \mu_A(1 \otimes \mu_A)$, and η is a bilateral unit $\eta : \Lambda \to A$, i.e.,



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Preliminaries Historical origin Mathematical notion

A_{∞} -structures: Historical evolution

The notion of A_{∞} -algebras appears in the literature as a generalization of "associative up to homotopy".

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 $\begin{array}{c} \text{Introduction}\\ A_{\infty}\text{-structures via perturbation}\\ \text{Theoretical study of complexity} \end{array}$

Preliminaries Historical origin Mathematical notion

A_{∞} -structures: Historical evolution

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Definition

Given M a differential graded module (DG-module), a morphism $\mu_2: M \otimes M \to M$ of degree zero is said **associative up to** homotopy if

 $\begin{array}{c} \text{Introduction}\\ A_{\infty}\text{-structures via perturbation}\\ \text{Theoretical study of complexity} \end{array}$

Preliminaries Historical origin Mathematical notion

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Preliminaries Historical origin Mathematical notion

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- it is compatible with the differential of M;
- there exists $\mu_3: M^{\otimes 3} \to M$ of degree +1, such that

 $\mu_2(\mu_2\otimes 1)-\mu_2(1\otimes \mu_2).$

Preliminaries Historical origin Mathematical notion

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- it is compatible with the differential of M;
- there exists $\mu_3: M^{\otimes 3} \to M$ of degree +1, such that

$$\mu_2(\mu_2\otimes 1)-\mu_2(1\otimes \mu_2)\neq 0.$$

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 $\begin{array}{c} \text{Introduction}\\ A_{\infty}\text{-structures via perturbation}\\ \text{Theoretical study of complexity} \end{array}$

Preliminaries Historical origin Mathematical notion

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 $\mu_3 d + d\mu_3 = \mu_2(\mu_2 \otimes 1) - \mu_2(1 \otimes \mu_2) \neq 0.$

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 $\begin{array}{c} \text{Introduction}\\ A_{\infty}\text{-structures via perturbation}\\ \text{Theoretical study of complexity} \end{array}$

Preliminaries Historical origin Mathematical notion

A_{∞} -structures: Classical Example

Let (X, *) be a topological space with a base point * and let ΩX denote the space of based loops in X: a point of ΩX is a continuous map $f : \mathbb{S}^1 \to X$ taking the base point of the circle to the base point *.

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Preliminaries Historical origin Mathematical notion

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Let us take as multiplication
the composition map
$$\mu_2: \Omega X \times \Omega X \to \Omega X$$
$$(f_1, f_2) \longrightarrow f_1 * f_2$$

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Preliminaries Historical origin Mathematical notion

A_{∞} -structures: Classical Example

Non associative

 μ_2 is not associative because of





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Preliminaries Historical origin Mathematical notion

A_{∞} -structures: Classical Example

Non associative





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Preliminaries Historical origin Mathematical notion

A_{∞} -structures: Classical Example



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Preliminaries Historical origin Mathematical notion

A_{∞} -structures: Mathematical notion

Definition

An A_{∞} -algebra is a graded module A, with a family of graded maps $m_i : A^{\otimes i} \to A$, of degree i - 2, such that for all $i \ge 1$:

$$\sum_{n=1}^{i}\sum_{k=0}^{i-n}(-1)^{n+k+nk}m_{i-n+1}(1^{\otimes k}\otimes m_n\otimes 1^{\otimes i-n-k})=0.$$

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Preliminaries Historical origin Mathematical notion

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• $m_1m_1 = 0 \Rightarrow m_1$ is a differential.

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Preliminaries Historical origin Mathematical notion

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• $m_1m_2 = m_2(m_1 \otimes 1 + 1 \otimes m_1) \Rightarrow m_2$ compatible with dif.

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Preliminaries Historical origin Mathematical notion

A_{∞} -structures: Mathematical notion

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• $m_1m_1 = 0 \Rightarrow m_1$ is a differential.

- $m_1m_2 = m_2(m_1 \otimes 1 + 1 \otimes m_1) \Rightarrow m_2$ compatible with dif.
- $m_3(m_1 \otimes 1^2 + 1^2 \otimes m_1 + 1 \otimes m_1 \otimes 1) + m_1 m_3 = m_2(m_2 \otimes 1 1 \otimes m_2).$

(日)

 $\begin{array}{c} \text{Introduction}\\ A_{\infty}\text{-structures via perturbation}\\ \text{Theoretical study of complexity} \end{array}$

Preliminaries Historical origin Mathematical notion

A_{∞} -structures: Mathematical notion

Example

Every DG-algebra is, in particular, an A_{∞} -algebra with $m_1 = d$, $m_2 = \mu$ and $m_i = 0$ for all $i \ge 3$.

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 $\begin{array}{c} \text{Introduction}\\ A_{\infty}\text{-structures via perturbation}\\ \text{Theoretical study of complexity} \end{array}$

Preliminaries Historical origin Mathematical notion

A_{∞} -structures: Mathematical notion

Example

Every DG-algebra is, in particular, an A_{∞} -algebra with $m_1 = d$, $m_2 = \mu$ and $m_i = 0$ for all $i \ge 3$.

Example

The chain complex of the loop space of X, $C_*(\Omega X)$ has an A_∞ -algebra structure.

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Contractions The tensor trick

Index

Introduction

- Preliminaries
- Historical origin
- Mathematical notion
- 2 A_{∞} -structures via perturbation
 - Contractions
 - The tensor trick
 - 3 Theoretical study of complexity
 - Main Results
 - Computational advantages

< 17 ▶

Contractions The tensor trick

Contractions

Definition

A contraction $c : \{N, M, f, g, \phi\}$ is a 5-tuple, such that

 (N, d_N) g (M, d_M)

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Contractions The tensor trick

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- $f: N_* \to M_*$ and $g: M_* \to N_*$ morphisms of degree zero;



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- $\phi: N_* \rightarrow N_{*+1}$ is a homotopy operator;

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(N, d_N)
g () f
(M, d_M)

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Contractions The tensor trick

Contractions

Definition

A contraction $c : \{N, M, f, g, \phi\}$ is a 5-tuple, such that

- (N, d_N) , (M, d_M) DG-modules.
- $f: N_* \rightarrow M_*$ and $g: M_* \rightarrow N_*$ morphisms of degree zero;
- $\phi: N_* \rightarrow N_{*+1}$ is a homotopy operator;
- $fg = 1_M$, $\phi d_N + d_N \phi + gf = 1_N$;

ϕ
(N, d_N)
g () f
(M, d_M)

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Contractions The tensor trick

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- $\phi: N_* \rightarrow N_{*+1}$ is a homotopy operator;

•
$$fg = 1_M$$
, $\phi d_N + d_N \phi + gf = 1_N$;

•
$$f\phi = 0$$
, $\phi g = 0$, $\phi \phi = 0$.

ϕ
(N, d_N)
g () f
(M, d_M)

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Contractions The tensor trick

Contractions: A_{∞} -structures

Theorem (Kad80,GLS91)

Let (A, d_A, μ) and (M, d_M) be a connected DG-algebra and a DG-module, respectively and $c : \{A, M, f, g, \phi\}$ a contraction between them. Then the DG-module M is provided with an A_{∞} -algebra structure

Contractions The tensor trick

Contractions: A_{∞} -structures

given by

$$m_n: M^{\otimes n} \to M$$

$$m_1 = -d_M$$

$$m_n = (-1)^{n+1} f \,\mu^{(1)} \,\phi^{[\otimes 2]} \,\mu^{(2)} \cdots \phi^{[\otimes n-1]} \,\mu^{(n-1)} \,g^{\otimes n} \,, \quad n \ge 2 \quad (1)$$

with

$$\mu^{(k)} = \sum_{i=0}^{k-1} (-1)^{i+1} 1^{\otimes i} \otimes \mu_{\mathsf{A}} \otimes 1^{\otimes k-i-1} ,$$
$$\phi^{[\otimes k]} = \sum_{i=0}^{k-1} 1^{\otimes i} \otimes \phi \otimes (g f)^{\otimes k-i-1}$$

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Contractions The tensor trick

Contractions: A_{∞} -structures

Computational Consequence

A contraction from a DG-algebra A to a DG-module M provides an algorithm to compute an A_{∞} -algebra structure on M.

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< 17 ▶

Index

Introduction

- Preliminaries
- Historical origin
- Mathematical notion
- 2 A_{∞} -structures via perturbation
 - Contractions
 - The tensor trick
- 3 Theoretical study of complexity
 - Main Results
 - Computational advantages

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Main Results

Theorem

Any composition of the kind $\phi^{[\otimes s]}\mu^{(s)}$ (s = 2, ..., n-1) in the formula (1), which is given by

$$\left(\sum_{j=0}^{s-1} 1^{\otimes j} \otimes \phi \otimes (g f)^{\otimes s-j-1}\right) \circ \left(\sum_{i=0}^{s-1} (-1)^{i+1} 1^{\otimes i} \otimes \mu_{\mathsf{A}} \otimes 1^{\otimes s-i-1}\right),$$

can be reduced to

$$\phi^{[\otimes s]}\mu^{(s)} = \sum_{i=0}^{s-1} (-1)^{i+1} 1^{\otimes i} \otimes \phi \mu_{\mathsf{A}} \otimes 1^{\otimes s-i-1}$$

Main Results Computational advantages

Main Results

$$\phi^{[\otimes s]}\mu^{(s)} = \sum_{i=0}^{s-1} (-1)^{i+1} 1^{\otimes i} \otimes \phi \mu_A \otimes 1^{\otimes s-i-1} .$$

$$(2)$$

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Main Results Computational advantages

Main Results

$$\phi^{[\otimes s]}\mu^{(s)} = \sum_{i=0}^{s-1} (-1)^{i+1} 1^{\otimes i} \otimes \phi \mu_A \otimes 1^{\otimes s-i-1}.$$
(2)

Moreover, given a composition of the kind

$$(\phi^{[\otimes s-1]}\mu^{(s-1)})\circ(\phi^{[\otimes s]}\mu^{(s)})$$
 $s=3,\ldots,n-2,$

for every index *i* in the sum (2) of $\phi^{[\otimes s]}\mu^{(s)}$, the formula of $\phi^{[\otimes s-1]}\mu^{(s-1)}$ in such a composition can be reduced to

$$\sum_{j=i-1,\,j\geq 0}^{s-2} (-1)^{j+1} 1^{\otimes j} \otimes \phi \mu_A \otimes 1^{\otimes s-j-2} \,. \tag{3}$$

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Main Results

In other words, the whole composition $(\phi^{[\otimes 2]} \mu^{(2)}) \circ \cdots \circ (\phi^{[\otimes n-1]} \mu^{(n-1)})$ in the formula of m_n can be expressed by

$$\sum_{i_{n-1}=0}^{n-2} \left(\cdots \left(\sum_{i_2=i_3-1}^{1} (\phi \mu)^{(2,i_2)} \right) \cdots \right) (\phi \mu)^{(n-1,i_{n-1})},$$

where $(\phi\mu)^{(k,j)} = (-1)^{j+1} \mathbb{1}^{\otimes j} \otimes \phi\mu_A \otimes \mathbb{1}^{\otimes k-j-1}$ and each addend exists whenever the corresponding index $i_k \geq 0$.

Main Results Computational advantages

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Scheme of the proof

$$m_{n} = (-1)^{n+1} f \mu^{(1)} \phi^{[\otimes 2]} \mu^{(2)} \cdots \phi^{[\otimes n-2]} \mu^{(n-2)} \underbrace{\phi^{[\otimes n-1]} \mu^{(n-1)} g^{\otimes n}}_{k=1}}_{k=2}$$

A. Berciano, M.J. Jiménez, P. Real On the computation of A_{∞} -maps

Main Results Computational advantages

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$$\boxed{k=1} \qquad \phi^{[\otimes n-1]} \mu^{(n-1)} g^{\otimes n}$$
$$\left(\sum_{j=0}^{n-2} 1^{\otimes j} \otimes \phi \otimes (gf)^{\otimes n-j-2}\right) \circ \left(\sum_{i=0}^{n-2} \pm g^{\otimes i} \otimes \mu_A g^{\otimes 2} \otimes g^{\otimes n-i-2}\right)$$

Main Results Computational advantages

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$$\begin{array}{l} \boxed{k=1} \qquad \phi^{[\otimes n-1]} \, \mu^{(n-1)} \, g^{\otimes n} \\ \\ \left(\sum_{j=0}^{n-2} 1^{\otimes j} \otimes \phi \otimes (gf)^{\otimes n-j-2}\right) \circ \left(\sum_{i=0}^{n-2} \pm g^{\otimes i} \otimes \mu_A \, g^{\otimes 2} \otimes g^{\otimes n-i-2}\right) \\ \\ \phi \, g = 0 \\ f \, g = 1 \end{array}$$

Main Results Computational advantages

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(日)

$$\begin{array}{c} \boxed{k=1} \qquad \phi^{[\otimes n-1]} \, \mu^{(n-1)} \, g^{\otimes n} \\ \\ \left(\sum_{j=0}^{n-2} 1^{\otimes j} \otimes \phi \otimes (gf)^{\otimes n-j-2}\right) \circ \left(\sum_{i=0}^{n-2} \pm g^{\otimes i} \otimes \mu_A \, g^{\otimes 2} \otimes g^{\otimes n-i-2}\right) \\ \\ \phi \, g = 0 \\ f \, g = 1 \qquad \Rightarrow \end{array}$$

Main Results Computational advantages

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< 日 > < 同 > < 三 > < 三 >

$$\begin{array}{ccc}
 k = 1 & \phi^{[\otimes n-1]} \, \mu^{(n-1)} \, g^{\otimes n} \\
 \left(\sum_{j=0}^{n-2} 1^{\otimes j} \otimes \phi \otimes (gf)^{\otimes n-j-2} \right) \circ \left(\sum_{i=0}^{n-2} \pm g^{\otimes i} \otimes \mu_A \, g^{\otimes 2} \otimes g^{\otimes n-i-2} \right) \\
 \phi \, g = 0 \\
 f \, g = 1 & \Rightarrow & \sum_{i=0}^{n-2} \pm 1^{\otimes i} \otimes \phi \, \mu_A \otimes 1^{\otimes n-i-2}
\end{array}$$

Main Results Computational advantages

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$$\left(\phi^{[\otimes n-2]} \mu^{(n-2)}\right) \circ \left(\sum_{i=0}^{n-2} \pm g^{\otimes i} \otimes \phi(sth.) \otimes g^{\otimes n-i-2}\right)$$

Main Results Computational advantages

э

$$k = 2$$

$$\left(\phi^{[\otimes n-2]} \mu^{(n-2)}\right) \circ \left(\sum_{i=0}^{n-2} \pm g^{\otimes i} \otimes \phi(sth.) \otimes g^{\otimes n-i-2}\right)$$

$$\phi g = 0$$

$$\phi \phi = 0$$

Main Results Computational advantages

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$$\begin{array}{c}
\hline k = 2 \\
\left(\phi^{[\otimes n-2]} \mu^{(n-2)}\right) \circ \left(\sum_{i=0}^{n-2} \pm g^{\otimes i} \otimes \phi(sth.) \otimes g^{\otimes n-i-2}\right) \\
\phi g = 0 \\
\phi \phi = 0 \qquad \Rightarrow
\end{array}$$

Main Results Computational advantages

э

< 日 > < 同 > < 三 > < 三 >

$$\begin{array}{l} \boxed{k=2} \\ \left(\phi^{[\otimes n-2]}\,\mu^{(n-2)}\right)\circ\left(\sum_{i=0}^{n-2}\pm g^{\otimes i}\otimes\phi(sth.)\otimes g^{\otimes n-i-2}\right) \\ \phi\,g=0 \\ \phi\,\phi=0 \qquad \Rightarrow \qquad \sum_{j=0}^{n-3}\pm 1^{\otimes j}\otimes\phi\mu_{A}\otimes(gf)^{\otimes n-j-3} \end{array}$$

Main Results Computational advantages

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< 日 > < 同 > < 三 > < 三 >

$$\begin{aligned} \boxed{k=2} \\ \left(\phi^{[\otimes n-2]} \mu^{(n-2)}\right) &\circ \left(\sum_{i=0}^{n-2} \pm g^{\otimes i} \otimes \phi(sth.) \otimes g^{\otimes n-i-2}\right) \\ \phi g &= 0 \\ \phi \phi &= 0 \end{aligned} \Rightarrow \qquad \sum_{j=0}^{n-3} \pm 1^{\otimes j} \otimes \phi \mu_A \otimes (gf)^{\otimes n-j-3} \\ f \phi &= 0 \end{aligned}$$

Main Results Computational advantages

Scheme of the proof

$$\begin{array}{l}
\boxed{k=2}\\
\begin{pmatrix}\phi^{[\otimes n-2]} \mu^{(n-2)} \end{pmatrix} \circ \left(\sum_{i=0}^{n-2} \pm g^{\otimes i} \otimes \phi(sth.) \otimes g^{\otimes n-i-2} \right)\\ \phi g = 0\\ \phi \phi = 0 \qquad \Rightarrow \qquad \sum_{j=0}^{n-3} \pm 1^{\otimes j} \otimes \phi \mu_A \otimes (gf)^{\otimes n-j-3}\\ f \phi = 0\\ \qquad \Rightarrow \end{array}$$

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Main Results Computational advantages

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$$\begin{array}{l}
\boxed{k=2}\\ \left(\phi^{[\otimes n-2]}\,\mu^{(n-2)}\right)\circ\left(\sum_{i=0}^{n-2}\pm g^{\otimes i}\otimes\phi(sth.)\otimes g^{\otimes n-i-2}\right)\\ \phi\,g=0\\ \phi\,\phi=0\\ \Rightarrow\\ f\,\phi=0\\ \Rightarrow\\ \hline\sum_{j=0}^{n-3}\pm 1^{\otimes j}\otimes\phi\mu_{A}\otimes(gf)^{\otimes n-j-3}\\ \sum_{j=i-1}^{n-3}\pm 1^{\otimes j}\otimes\phi\mu_{A}\otimes 1^{\otimes n-j-3}\\ \end{array}\right)$$

Main Results Computational advantages

Initial formulation

$$m_n = (-1)^{n+1} f \, \mu^{(1)} \, \phi^{[\otimes 2]} \, \mu^{(2)} \cdots \phi^{[\otimes n-1]} \, \mu^{(n-1)} \, g^{\otimes n} \,, \quad n \geq 2 \,.$$

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Main Results Computational advantages

Initial formulation

$$m_n = (-1)^{n+1} f \, \mu^{(1)} \, \phi^{[\otimes 2]} \, \mu^{(2)} \cdots \phi^{[\otimes n-1]} \, \mu^{(n-1)} \, g^{\otimes n} \,, \quad n \geq 2 \,.$$

•
$$\phi^{[\otimes k]} = \sum_{i=0}^{k-1} 1^{\otimes i} \otimes \phi \otimes (g f)^{\otimes k-i-1} \to k$$
 addends.

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Main Results Computational advantages

Initial formulation

$$m_n = (-1)^{n+1} f \, \mu^{(1)} \, \phi^{[\otimes 2]} \, \mu^{(2)} \cdots \phi^{[\otimes n-1]} \, \mu^{(n-1)} \, g^{\otimes n} \,, \quad n \geq 2 \,.$$

•
$$\phi^{[\otimes k]} = \sum_{i=0}^{k-1} 1^{\otimes i} \otimes \phi \otimes (g f)^{\otimes k-i-1} \to k$$
 addends.
• $\mu^{(k)} = \sum_{i=0}^{k-1} (-1)^{i+1} 1^{\otimes i} \otimes \mu_A \otimes 1^{\otimes k-i-1} \to k$ addends.

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Main Results Computational advantages

Initial formulation

$$m_n = (-1)^{n+1} f \, \mu^{(1)} \, \phi^{[\otimes 2]} \, \mu^{(2)} \cdots \phi^{[\otimes n-1]} \, \mu^{(n-1)} \, g^{\otimes n} \,, \quad n \geq 2 \,.$$

•
$$\phi^{[\otimes k]} = \sum_{i=0}^{k-1} 1^{\otimes i} \otimes \phi \otimes (g f)^{\otimes k-i-1} \to k$$
 addends.

•
$$\mu^{(k)} = \sum_{i=0}^{\kappa-1} (-1)^{i+1} 1^{\otimes i} \otimes \mu_A \otimes 1^{\otimes k-i-1} \to k$$
 addends.

• $\phi^{[\otimes k]} \mu^{(k)}$ contributes with k^2 addends.

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Main Results Computational advantages

Initial formulation

$$m_n = (-1)^{n+1} f \, \mu^{(1)} \, \phi^{[\otimes 2]} \, \mu^{(2)} \cdots \phi^{[\otimes n-1]} \, \mu^{(n-1)} \, g^{\otimes n} \,, \quad n \geq 2 \,.$$

•
$$\phi^{[\otimes k]} = \sum_{i=0}^{k-1} 1^{\otimes i} \otimes \phi \otimes (g f)^{\otimes k-i-1} \to k$$
 addends.
• $\mu^{(k)} = \sum_{i=0}^{k-1} (-1)^{i+1} 1^{\otimes i} \otimes \mu_A \otimes 1^{\otimes k-i-1} \to k$ addends.

•
$$\phi^{[\otimes k]} \, \mu^{(k)}$$
 contributes with k^2 addends.

•
$$m_n$$
 is $O((n-1)!^2)$ in space.

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Main Results Computational advantages

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Initial formulation

$$m_n = (-1)^{n+1} f \, \mu^{(1)} \, \phi^{[\otimes 2]} \, \mu^{(2)} \cdots \phi^{[\otimes n-1]} \, \mu^{(n-1)} \, g^{\otimes n} \,, \quad n \geq 2 \,.$$

•
$$\phi^{[\otimes k]} = \sum_{i=0}^{k-1} 1^{\otimes i} \otimes \phi \otimes (g f)^{\otimes k-i-1} \to k$$
 addends.

•
$$\mu^{(k)} = \sum_{i=0}^{k} (-1)^{i+1} 1^{\otimes i} \otimes \mu_A \otimes 1^{\otimes k-i-1} \to k$$
 addends.

•
$$\phi^{[\otimes k]} \mu^{(k)}$$
 contributes with k^2 addends.

•
$$m_n$$
 is $O((n-1)!^2)$ in space.

The number of basic operations can be expressed by $n + n(n-1)!^2 + \frac{(n+3)(n-2)}{4}(n-1)!^2$, so m_n is $O((n)!^2)$ in time.

Main Results Computational advantages

First reduction

$$\phi^{[\otimes k]}\mu^{(k)} = \sum_{i=0}^{k-1} (-1)^{i+1} 1^{\otimes i} \otimes \phi \mu_{\mathsf{A}} \otimes 1^{\otimes k-i-1} \,.$$

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Main Results Computational advantages

First reduction

$$\phi^{[\otimes k]}\mu^{(k)} = \sum_{i=0}^{k-1} (-1)^{i+1} 1^{\otimes i} \otimes \phi \mu_{\scriptscriptstyle A} \otimes 1^{\otimes k-i-1}$$

$$\phi^{[\otimes k]} \mu^{(k)}$$
: k^2 addends $\rightarrow k$ addends.
So, now m_n is $O((n-1)!)$ in space.

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Main Results Computational advantages

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First reduction

$$\phi^{[\otimes k]}\mu^{(k)} = \sum_{i=0}^{k-1} (-1)^{i+1} 1^{\otimes i} \otimes \phi \mu_{\mathsf{A}} \otimes 1^{\otimes k-i-1}$$

$$\phi^{[\otimes k]} \mu^{(k)}$$
: k^2 addends $\rightarrow k$ addends.
So, now m_n is $O((n-1)!)$ in space.

Now, the number of operations is exactly

$$n+(n-1)!(2n-2)$$
,

so, m_n is O((n)!) in time.

Main Results Computational advantages

Second reduction

$$\phi^{[\otimes k]}\mu^{(k)} = \sum_{j=i-1, j\geq 0}^{k-2} (-1)^{j+1} 1^{\otimes j} \otimes \phi\mu_A \otimes 1^{\otimes k-j-2} \,.$$

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Main Results Computational advantages

Second reduction

$$\phi^{[\otimes k]}\mu^{(k)} = \sum_{j=i-1, j\geq 0}^{k-2} (-1)^{j+1} 1^{\otimes j} \otimes \phi\mu_A \otimes 1^{\otimes k-j-2}.$$

The number of addends becomes $(n-1)! - S_n$,

$$\frac{(n-1)!}{2} < (n-1)! - S_n < (n-1)!,$$

But $(n-1)! - S_n$ is much "closer" to $\frac{(n-1)!}{2}$ than to (n-1)!:

Main Results Computational advantages

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Second reduction

$$\phi^{[\otimes k]}\mu^{(k)} = \sum_{j=i-1,j\geq 0}^{k-2} (-1)^{j+1} 1^{\otimes j} \otimes \phi\mu_A \otimes 1^{\otimes k-j-2}.$$

The number of addends becomes $(n-1)! - S_n$,

$$\frac{(n-1)!}{2} < (n-1)! - S_n < (n-1)!,$$

But $(n-1)! - S_n$ is much "closer" to $\frac{(n-1)!}{2}$ than to (n-1)!:

п	5	10	50	100
$((n-1)! - S_n)/(n-1)!$	0,70833	0,5637	0,51042	0,5051

Main Results Computational advantages

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Comparative table

Summing up,

	original formula		new formula	
	time	space	time	space
m _n	$O(n!^2)$	$O((n-1)!^2)$	<i>O</i> (<i>n</i> !)	O((n-1)!)