Generalized Chebyshev polynomials and plane trees

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Abstract

The paper considers the notion of generalized Chebyshev polynomials. Their connection with plane trees and Galois theory is described. Belyi functions are defined as a further generalization of generalized Chebyshev polynomials.

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1 Introduction

1.1 P. Chebyshev

Pafnuty Lvovich Chebyshev (May 26, 1821 – December 8, 1894) was one of nine children who was born in the village of Okatovo, Borovsk District, Kaluga province into the family of the landowner Lev Pavlovich Chebyshev. In 1832 the family moved to Moscow. In 1837 Chebyshev started the studies of mathematics at the philosophical department of Moscow University and graduated from the university as "the most outstanding candidate".

In 1847, Chebyshev defended his dissertation "About integration with the help of logarithms" at St Petersburg University. Chebyshev lectured at the university from 1847 to 1882. In 1882 he left the university and completely devoted his life to research. Chebyshev is known for his work in the field of probability, statistics and number theory. Chebyshev is considered to be one of the founding fathers of Russian mathematics. Among his students were Aleksandr Lyapunov and Andrey Markov.

1.2 Chebyshev polynomials

The notion of Chebyshev polynomials is well known. They are useful in various fields of studies. A lot of generalizations of them have been found and investigated. In this paper we shall talk about one of them, which can also be referred to as Shabat polynomials. Here we shall discuss one-to-one correspondence of Shabat polynomials, plane trees and finite extensions of rational numbers.

2 Basic Definitions

In this section we shall give several definitions of Chebyshev polynomials. Then we turn to motivation of definition of Shabat polynomials, and finally we shall give the formal definition of these polynomials.

2.1 Definitions and properties of Chebyshev polynomials

The most widely used definition of Chebyshev polynomials can be given by the following formula:

$$T_n(\cos(x)) = \cos(nx)$$

This definition is useful for calculations with Chebyshev polynomials. For example, the following property can be easily proved:

Theorem 1 $T_n(T_m(x)) = T_{mn}(x) = T_m(T_n(x))$

Proof

$$T_n(T_m(x)) = T_n(\cos(mx)) = \cos(mnx) = T_{mn}(\cos(x))$$

□

Chebyshev polynomials can also be defined With the help of the following norm:

Definition 2.1 *Let $f(x)$ be a continuous function defined on the segment $[-1, 1]$. Then we define the norm of this function $|f| = max(f(x)|x \in [-1, 1])$.*

Then the definition of Chebyshev polynomials can be given in the following way:

Definition 2.2 *$T_n(x)$ is the polynomial with the smallest norm among all the polynomials with the leading coefficient $\frac{1}{2^n}$.*

Note that Chebyshov Polynomials reach their maximal value exactly in $n+1$ points which are $\cos(\frac{k}{n}\pi)$ where $k = 0 \dots n$.

This property of Chebyshev polynomials is important in the approximation theory. The roots of the Chebyshev polynomials (which are also referred to as Chebyshev nodes) are theoretically the best nodes functions with polynomials.

The following theorem gives us one more property of Chebyshev polynomials, which is also sometimes used as their definition.

Theorem 2 *Chebyshev polynomials form an orthonormal basis for the space of polynomials in one variable with respect to the scalar product defined in the following way:*

$$< f, g >= \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1+x^2}}.$$

For conclusion we give here a number of properties which can be used to calculate the coefficients of Chebyshev polynomials directly.

Theorem 3 *The recursive formula for Chebyshev polynomials*

$$T_{n+1}(x) = 2xT_n(x) + T_{n-1}(x)$$

And two direct formulas

$$T_n(x) = \sum_{2j \leq n} \binom{n}{2j} x^{n-2j}(x^2-1)^j$$

$$T_n(x) = \frac{(-1)^n\sqrt{1-x^2}}{(2n-1)!!} \frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}}$$

2.2 Motivation of definition of generalized Chebyshev polynomials

Consider a polynomial $P(z)$ of degree n with complex coefficients. It maps a complex plane onto another one. Take point $w \in \mathbb{C}$ and consider its inverse image $P^{-1}(w) = \{z|P(z) = w\}$. This set usually consists of n distinct points, but if z is also a root of $P'(z)$ then the size of this set is smaller than its usual value.

Definition 2.3 *A complex number z is called a critical point of P if* $P'(z) = 0$

Definition 2.4 *Critical value of polynomial is its value in its critical point.*

Definition 2.5 *Let z be a critical point of P and let $P'(z) = P''(z) = \cdots = P^{(k-1)}(z) = 0$ and $P^{(k)}(z) \neq 0$ then we say that critical point z has order k. Also sometimes we shall call non-critical points critical points of order* 1.

Since $P'(z)$ is a polynomial of degree $n-1$, the polynomial $P(z)$ usually has $n-1$ critical points(of order 2) and $n-1$ critical values. But in some cases critical points glue together forming a critical point of a higher order and/or some critical values may be attached into the same value. What we are interested in is the most degenerate case, when the set of critical values of $P(z)$ is as small as possible.

It can be easily seen that the set of critical values of polynomial consists of only one critical value if and only if this polynomial is z^n or its linear transformation.

And now we can define generalized Chebyshev polynomials.

Definition 2.6 *Polynomial $f(z)$ is called generalized Chebyshev polynomial (GCP) if it has two critical values at most.*

We immediately see two simple examples which are z^n and $T_n(z)$.

The other definition of GCP can be obtained from the following theorem.

Theorem 4 *Let w_1 and w_2 be distinct complex numbers. Then for each polynomial $P(z)$ the size of the union of $P^{-1}(w_1)$ and $P^{-1}(w_2)$ is at least $n+1$. And this maximum is reached if and only if P is GCP.*

Proof Let

$$P(z) - w_1 = \prod_{i=1}^{d_1} (z - a_i)^{\alpha_i}$$

$$P(z) - w_2 = \prod_{i=1}^{d_2} (z - b_i)^{\beta_i}$$

where a_i and b_i are distinct complex numbers. Then we have the following:

$$\prod_{i=1}^{d_1} (z - a_i)^{\alpha_i - 1} \prod_{i=1}^{d_2} (z - b_i)^{\beta_i - 1} | P'(z)$$

And we see that $\sum_{i=1}^{d_1} \alpha_i - 1 + \sum_{i=1}^{d_2} \beta_i - 1 \leq deg(P'(z)) = n-1$. But $\sum_{i=1}^{d_1} \alpha_i = \sum_{i=1}^{d_2} \beta_i = deg(P(z)) = n$, so we got $n - d_1 + n - d_2 \leq n - 1$ and finally $d_1 + d_2 \geq n + 1$. Here $d_1 + d_2$ equals the size of the union of inverse images of w_1 and w_2. So the inequality in question is proved. The equality takes place if and only if $P'(z)$ can be factored into the product of $z - a_i$ and $z - b_i$ in certain powers. But it means that the only critical values of P are the values of P in points a_i and b_i, which are w_1 and w_2 and the theorem is proved. □

It can be easily seen that linear transformations of GCP are also GSP. More formally: if L_1, L_2 are linear functions $\mathbb{C} \to \mathbb{C}$, and $f(z)$ is GCP, then $L_1(f(L_2(z))))$ is also a GCP. Now we can introduce an equivalence on GCP: polynomials $f(z)$ and $g(z)$ are called equivalent if one of them is a linear transformation of another one ($\exists L_1, L_2 : g(z) = L_1(f(L_2(z)))))$.

2.3 Combinatorial bicolored plane trees

Everyone knows that a tree is a connected graph without circuits. Each tree can be drawn on the plane in a number of ways. We are going to introduce an additional structure on a tree to fix the type of picture of this tree.

Definition 2.7 *Plane tree is a structure of a tree and for every vertex v a cyclic order on the vertices adjacent to v.*

So in this definition for each vertex we fix the order in which the edges adjacent to it should be drawn. Now we shall define what plane trees we should consider equal.

Definition 2.8 *Plane trees are called isomorphic if there exists an isomorphism of trees as graphs which also preserves cyclic permutations from the definition of plane trees.*

The equivalence class of plane trees is called a combinatorial plane tree.

Every tree has a natural structure of a bipartite graph: its vertices can be colored in two colors. So through fixing one of the two colorings we obtain a bicolored plane tree.

Definition 2.9 *Bicolored plane tree is a plane tree with vertices colored with two colors, in a way described above.*

In the same natural way as in the case of plane trees, we shall introduce an isomorphism of bicolored plane trees. Classes of equivalence of bicolored plane trees are combinatorial bicolored plane trees. For the sake of brevity from now on the term tree means a combinatorial bicolored plane tree.

3 Correspondence between trees and Chebyshev polynomials

3.1 Definition

Let us consider the inverse image of a segment. We colored the ends of the segment in black and white colors. Let $[C_0, C_1]$ be the segment free of critical values of polynomial $P(z)$. Then the inverse image would be a set of disjoint sets, each of them being homeomorphic to a segment.

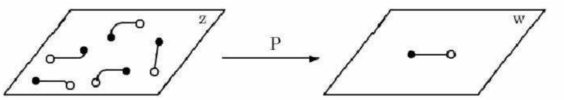


Figure 1: Inverse image of a segment

Then we suppose that there are no critical points inside a segment, but they may appear at one or both ends. Then several curvilinear segments may glue together with the monochrome vertexes.

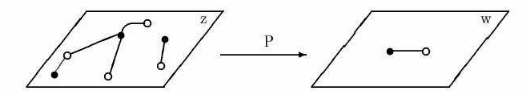


Figure 2: Inverse image of a segment with critical values on its ends.

Anyway we get a bicolored graph on a plane, whose vertices are inverse images of the ends of segment.

Let us consider the case when $P(z)$ is GCP and the segment joins its critical values. First we look at two simple examples: $P(z) = z^n$ Its only critical value is 0 and we choose the second point to be equal 1. So we have a "star" since the solutions of equality $z^n = t$ for $t \in [0, 1]$ have the argument equal to $\frac{k}{n}2\pi$.

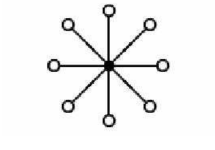


Figure 3: $P(z) = z^n$.

Another example is $P(z) = T_n(z)$. Its critical values are 1 and -1. The inverse image of the segment $[-1, 1]$ consists of only real numbers. So we get a "chain".

Figure 4: $P(z) = T_n(z)$.

Now given a GCP, we can construct a bicolored plane graph. The following theorem gives us more information on the graph constructed.

Theorem 5 *For each GCP $f(z)$ the graph constructed is a bicolored plane tree.*

Proof It is clear that the constructed graph has $n+1$ vertices (since the union of inverse images of critical values of GCP consists of $n+1$ points) and n edges (because the segment has no critical values on it, and the inverse image of each point on it consists on exactly n points). What we need is to prove that the graph has no circuits.

In case of existence of a circuit, our polynomial $f(z)$ takes only real values on the boundary of domain bounded by our circuit. It means that the harmonic function $Im(f(z))$ equals zero on the boundary. But it means that $Im(f(z)) = 0$ on the whole domain, which is a contradiction. □

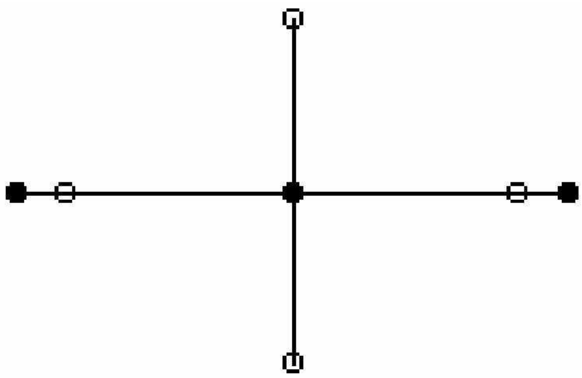
And now the central theorem of this section.

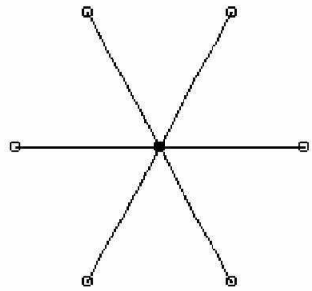
Theorem 6 *For each combinatorial bicolored plane tree T there exists a GCP such that the corresponding inverse image is isomorphic to T. Moreover, this GCP is unique up to the equivalence introduced above.*

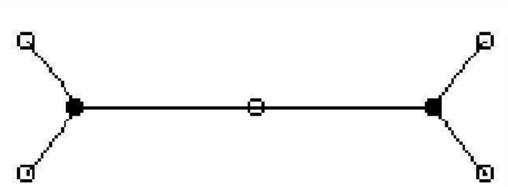
And we have a bijection between the classes of equivalence of GCP and combinatorial bicolored plane trees.

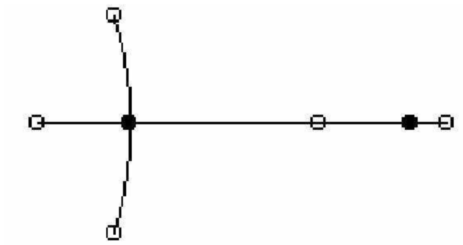
3.2 Canonical geometric form

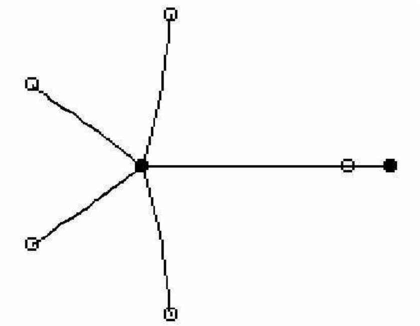
As we know from the previous section, for each tree we can construct a GCP. Also, given a GCP, we are able to construct a picture of a tree on the complex plane and the view of this picture is unique up to linear transformations of complex plane which do not change the geometric form of the picture. So for each tree we have a canonical way how to draw it on the plain. Now we shall look at several pictures of canonical geometric forms trees.

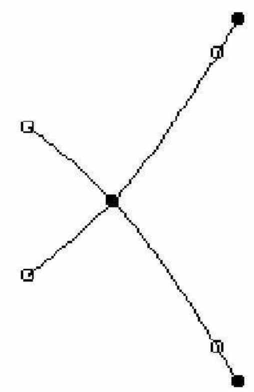


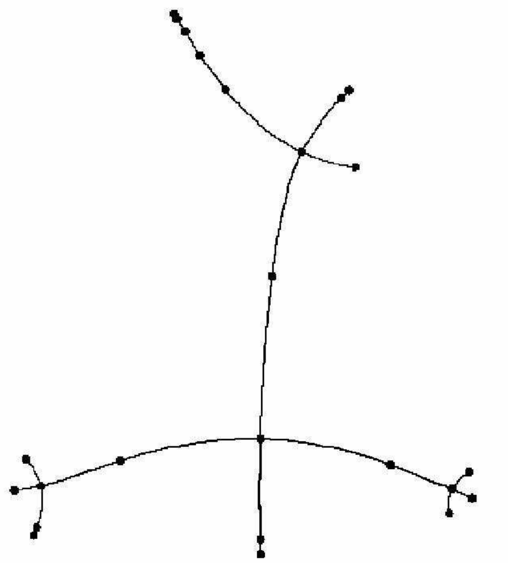












3.3 Calculation of GCP

The main question of the section is: given a tree, how can we calculate the coefficients of the corresponding GCP? The first way is based on the following system of polynomial equations.

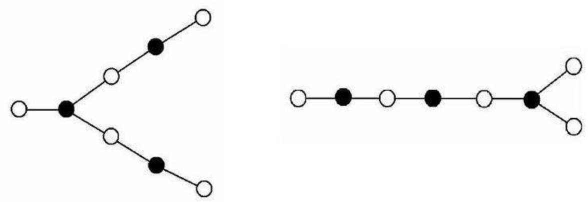
$$P(z) = \lambda \prod_{i=1} d_1 (z - a_i)^{\alpha_i}$$

$$P(z) - 1 = \lambda \prod_{i=1} d_2 (z - b_i)^{\beta_i}$$

Here we suppose critical values of P to be 0 and 1. α_i and β_i are the degrees of black and white vertices of the tree. These two polynomial equations give us n algebraic equations in $n+2$ variables. It allows us to find coefficients of GCP since these two degrees of freedom correspond to the choice of linear transformation.

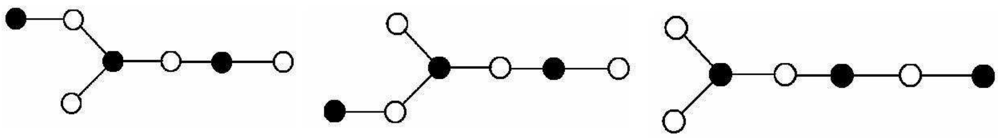
When we calculate coefficients of GCP, we use only the degrees of vertices of our tree. So for a number of trees we have the same system of equation. All such trees form a family $<\alpha, \beta>$ where $\alpha = (\alpha_1, \alpha_2, \dots)$, $\beta = (\beta_1, \beta_2, \dots)$. By solving the system of equations we can find all the polynomials which correspond to the trees of this family.

Here are examples of trees and their polynomials:



$$P(z) = z^3(z^2 - 2z + a)$$

where $a = \frac{34+6\sqrt{21}}{7}$ for the left tree and $a = \frac{34-6\sqrt{21}}{7}$ for the right one.



$$P(z) = z^3(z-1)^2(z-a)$$

where a is a root of polynomial $25a^3 - 12a^2 - 24a - 16$.

The computation of coefficients of GCP for a tree with a big number of vertices becomes a complicated problem since we have to solve a system of algebraic equations of a high degree, so another way is to be found, and it has been found. The idea of the new method is the following:

First we calculate coefficients not precisely. It can be done by using the class of polynomials with the following conditions: 1.There exist complex numbers C and w such that $g(w) = C$, $f'(w) = 0$, $f''(w) \neq 0$

2.If $f'(z) = 0$, then $f(z) = \pm 1$ or $z = w$.

By changing the parameters C and w we can find the approximate values of coefficients of each GCP.

And in the second stage we find precise values of coefficients.

Here we give a small example of GCP which has been calculated with this method, but cannot be calculated in any other way.

$$q_{11}z^{11} + \sum_{k=3}^{9} q_k z^k + z^2 + z$$

$$q_3 = -\frac{133834873}{160133408} + \frac{15160167\sqrt{-11}}{160133408}$$

$$q_4 = -\frac{5232924722141}{17910921684800} + \frac{627451817123\sqrt{-11}}{17910921684800}$$

$$q_5 = \frac{5886549671704763 1}{24040039085338 5600} - \frac{29406421083302111\sqrt{-11}}{480800781706771200}$$

$$q_6 = \frac{20517898169187936 68639}{672219592923779484 00000} - \frac{6443822594602867342 67\sqrt{-11}}{672219592923779484 00000}$$

$$q_7 = -\frac{881040112562681914516311 51}{32080111559903887055104 00000} + \frac{44903432982731017638427353\sqrt{-11}}{3208011155990388705510400000}$$

$$q_8 = -\frac{12635455258870878123168485 4967}{2152896286785149860268029440 00000} + \frac{14459655258391192167641093 8701\sqrt{-11}}{2152896286785149860268029440 00000}$$

$$q_9 = \frac{73319165764050208397831570887 4249}{72240434903075703561293727859 2000000} - \frac{79420812869387101237958661853 18901\sqrt{-11}}{57792347922460562849034982287 36000000}$$

$$q_{11} = \frac{465668733986203040846410356648078889132 18531}{21208196233458047170990786140383440281600 00000000} +$$

$$+\frac{10207079650537898540933872108962518640173 1007\sqrt{-11}}{21208196233458047170990786140383440281600 00000000}$$

4 GCP in Galois theory

It can be easily seen that the group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the set of GCP with algebraic coefficients since their coefficients are solutions of a system of algebraic equations with rational coefficients. So Galois group moves GCP to GCP. This leads us to the concept of the field of definition of a tree. So we have defined the action of Galois group on trees (since there is a one-to-one correspondence between trees and GCP).

Definition 4.1 *Field of definition of a tree is a field which corresponds to the subgroup of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ that fixes the specific tree.*

The next theorem makes it clear that this notion is really useful.

Theorem 7 *For any bicolored plane tree there exists a generalized Chebyshev polynomial whose coefficients belong to the field of definition of the tree. Moreover, each finite extension of rational numbers is a field of definition of certain tree.*

5 Composition of GCP

It is well known that the composition of Chebyshev polynomials is Chebyshev polynomial, too. And is this the case for generalized Chebyshev polynomials? The next theorem gives an answer.

Theorem 8 *Let f, g be GCPwith critical values equal to 0 and 1 such that $/f(0), f(1)/$ lie in $/0, 1/$. Then the composition $f(g(x))$ is also GCP.*

Proof $f(g(x))' = 0 \Rightarrow g'(x)f'(g(x)) = 0 \Rightarrow g'(x) = 0$ or $f'(g(x)) = 0 \Rightarrow f(\{0,1\}) \subset \{0,1\}$ $\square$

Let us imagine that we have two trees T_P and T_G. Their polynomials are $P(z)$ and $G(z)$. And we need to construct a tree which corresponds to $P(G(z))$. We can calculate $P(z)$ and $G(z)$, find their composition and then construct a tree.But there is a direct way to do it.

First note that the geometrical meaning of condition $F(\{0,1\}) \subset \{0,1\}$ is that there are two vertices of T_P that lie at the points 0 and 1. Thus, T_P is not simply a tree but a tree with two distinct vertices. These two vertices divide T_P into 3 parts: "body", "head" and "tail". Where the "body" contains the vertices between distinguished ones, including them, the "head" contains vertices from the side of 0 including 0 and the "tail" contains vertices from the side of 1 including 1. And finally, in T_P we substitute the "body" for each edge (in such a way that zero vertices from "bodies" glues to white vertices of T_Q), and then for each vertex of T_Q we glue the number of "heads" or "tail" with respect to the color of this vertex to this vertex in such a way that there is exactly one "head" ("tail") between each two "bodies". This completes the construction.

6 Belyi functions

Given a tree, we know how to construct a polynomial. But we also can find a solution in the case of plane maps.

Definition 6.1 *Let f be a rational function on the Riemann sphere that satisfies the following conditions: 1. f has only three critical values: 0, 1, ∞ 2. All points of $f^{-1}(1)$ are critical with the degree exactly equal to 2. These functions are called Belyi functions.*

The idea is that for each Belyi function we can construct a plane map, and this correspondence is a bijection if Belyi functions are considered up to linear transformations of Riemann sphere(i.e. functions of type $\frac{az+b}{cz+d}$).

The map construction procedure from Belyi function looks as follows:

$f^{-1}(\infty)$ are vertices.

$f^{-1}(1,\infty)$ are edges ($f^{-1}(1)$ are points on the edges)

$f^{-1}(0)$ are points on faces of a map.

And the last theorem shows the connection between GCP and Belyi functions.

Theorem 9 *Let $P(z)$ be a GCP with critical values equal to -1 and 1. T_P is the tree which corresponds to P. Then the Belyi function which corresponds to T_P can be calculated with the following formula:*

$$f(z) = \frac{1}{1 - P(z)^2}$$

References

[1] G.Shabat, A.Zvonkin. Plane Trees and Algebraic Numbers: `www.labri.fr/perso/zvonkin/Research/shabzvon.ps.gz`.

[2] `http://logic.pdmi.ras.ru/~yumat/Journal/Chebyshev/chebysh.htm`.