6.3 The Characteristic Polynomial

Consider the recurrence relation:

 $c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$

This is the general form of a linear recurrence relation of order k with constant coefficients ($c_0, c_k \neq 0$).

- T(n) only depends on the k preceding values. This means the recurrence relation is of order k.
- ► The recurrence is linear as there are no products of *T*[*n*]'s.
- ► If f(n) = 0 then the recurrence relation becomes a linear, homogenous recurrence relation of order k.

Note that we ignore boundary conditions for the moment.

6.3 The Characteristic Polynomial

Observations:

- ► The solution T[1], T[2], T[3],... is completely determined by a set of boundary conditions that specify values for T[1],...,T[k].
- In fact, any k consecutive values completely determine the solution.
- k non-concecutive values might not be an appropriate set of boundary conditions (depends on the problem).

Approach:

- First determine all solutions that satisfy recurrence relation.
- Then pick the right one by analyzing boundary conditions.
- First consider the homogenous case.

The solution space

$$S = \left\{ \mathcal{T} = T[1], T[2], T[3], \dots \mid \mathcal{T} \text{ fulfills recurrence relation} \right\}$$

is a vector space. This means that if $\mathcal{T}_1, \mathcal{T}_2 \in S$, then also $\alpha \mathcal{T}_1 + \beta \mathcal{T}_2 \in S$, for arbitrary constants α, β .

How do we find a non-trivial solution?

We guess that the solution is of the form λ^n , $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \dots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \ge k$.

Dividing by λ^{n-k} gives that all these constraints are identical to

$$\underbrace{c_0 \lambda^k + c_1 \lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \dots + c_k}_{\text{characteristic polynomial } P[\lambda]} = 0$$

This means that if λ_i is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, ..., \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_k\lambda_k^n$$

is a solution for arbitrary values α_i .

Lemma 1

Assume that the characteristic polynomial has k distinct roots $\lambda_1, \ldots, \lambda_k$. Then all solutions to the recurrence relation are of the form

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_k\lambda_k^n$$
.

Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \ldots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.



Proof (cont.).

Suppose I am given boundary conditions T[i] and I want to see whether I can choose the $\alpha'_i s$ such that these conditions are met:

$$\begin{array}{rcl} \alpha_{1} \cdot \lambda_{1} & + & \alpha_{2} \cdot \lambda_{2} & + & \cdots & + & \alpha_{k} \cdot \lambda_{k} & = & T[1] \\ \alpha_{1} \cdot \lambda_{1}^{2} & + & \alpha_{2} \cdot \lambda_{2}^{2} & + & \cdots & + & \alpha_{k} \cdot \lambda_{k}^{2} & = & T[2] \\ & & & \vdots \\ \alpha_{1} \cdot \lambda_{1}^{k} & + & \alpha_{2} \cdot \lambda_{2}^{k} & + & \cdots & + & \alpha_{k} \cdot \lambda_{k}^{k} & = & T[k] \end{array}$$



Proof (cont.).

Suppose I am given boundary conditions T[i] and I want to see whether I can choose the $\alpha'_i s$ such that these conditions are met:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ & \vdots & & \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_k^k \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} T[1] \\ T[2] \\ \vdots \\ T[k] \end{pmatrix}$$

We show that the column vectors are linearly independent. Then the above equation has a solution.



$$\begin{vmatrix} \lambda_{1} & \lambda_{2} & \cdots & \lambda_{k-1} & \lambda_{k} \\ \lambda_{1}^{2} & \lambda_{2}^{2} & \cdots & \lambda_{k-1}^{2} & \lambda_{k}^{2} \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_{1}^{k} & \lambda_{2}^{k} & \cdots & \lambda_{k-1}^{k} & \lambda_{k}^{k} \end{vmatrix} = \prod_{i=1}^{k} \lambda_{i} \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_{1} & \lambda_{2} & \cdots & \lambda_{k-1} & \lambda_{k} \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_{1}^{k-1} & \lambda_{2}^{k-1} & \cdots & \lambda_{k-1}^{k-1} & \lambda_{k}^{k-1} \end{vmatrix}$$
$$= \prod_{i=1}^{k} \lambda_{i} \cdot \begin{vmatrix} 1 & \lambda_{1} & \cdots & \lambda_{1}^{k-1} & \lambda_{k} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{1}^{k-1} & \lambda_{2}^{k-1} & \cdots & \lambda_{k-1}^{k-1} & \lambda_{k}^{k-1} \end{vmatrix}$$



$$\begin{vmatrix} \lambda_{1} & \cdots & \lambda_{1}^{k-2} & \lambda_{1}^{k-1} \\ 1 & \lambda_{2} & \cdots & \lambda_{2}^{k-2} & \lambda_{2}^{k-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_{k} & \cdots & \lambda_{k}^{k-2} & \lambda_{k}^{k-1} \end{vmatrix} = \\ \begin{vmatrix} 1 & \lambda_{1} - \lambda_{1} \cdot 1 & \cdots & \lambda_{1}^{k-2} - \lambda_{1} \cdot \lambda_{1}^{k-3} & \lambda_{1}^{k-1} - \lambda_{1} \cdot \lambda_{1}^{k-2} \\ 1 & \lambda_{2} - \lambda_{1} \cdot 1 & \cdots & \lambda_{2}^{k-2} - \lambda_{1} \cdot \lambda_{2}^{k-3} & \lambda_{2}^{k-1} - \lambda_{1} \cdot \lambda_{2}^{k-2} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_{k} - \lambda_{1} \cdot 1 & \cdots & \lambda_{k}^{k-2} - \lambda_{1} \cdot \lambda_{k}^{k-3} & \lambda_{k}^{k-1} - \lambda_{1} \cdot \lambda_{k}^{k-2} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix} = \\ \begin{bmatrix} k \\ \prod_{i=2}^k (\lambda_i - \lambda_1) \cdot \\ \vdots & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \\ \vdots & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$



Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^k \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all λ_i 's are different, then the determinant is non-zero.



What happens if the roots are not all distinct?

Suppose we have a root λ_i with multiplicity (Vielfachheit) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^n$. To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0 \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \dots + c_k \lambda^{n-k}$$

Since λ_i is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root λ_i .

This means

$$c_0 n \lambda_i^{n-1} + c_1 (n-1) \lambda_i^{n-2} + \cdots + c_k (n-k) \lambda_i^{n-k-1} = 0$$

Hence,

$$c_0 \underbrace{n\lambda_i^n}_{T[n]} + c_1 \underbrace{(n-1)\lambda_i^{n-1}}_{T[n-1]} + \cdots + c_k \underbrace{(n-k)\lambda_i^{n-k}}_{T[n-k]} = 0$$



Suppose λ_i has multiplicity *j*. We know that

$$c_0 n \lambda_i^n + c_1 (n-1) \lambda_i^{n-1} + \dots + c_k (n-k) \lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with λ ; plugging in λ_i)

Doing this again gives

$$c_0 n^2 \lambda_i^n + c_1 (n-1)^2 \lambda_i^{n-1} + \dots + c_k (n-k)^2 \lambda_i^{n-k} = 0$$

We can continue j - 1 times.

Hence, $n^{\ell} \lambda_i^n$ is a solution for $\ell \in 0, ..., j-1$.

Lemma 2

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0T[n] + c_1T[n-1] + \cdots + c_kT[n-k] = 0$$

Let λ_i , i = 1, ..., m be the (complex) roots of $P[\lambda]$ with multiplicities ℓ_i . Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i-1} \alpha_{ij} \cdot (n^j \lambda_i^n) .$$

The full proof is omitted. We have only shown that any choice of α_{ij} 's is a solution to the recurrence.

Example: Fibonacci Sequence

$$T[0] = 0$$

 $T[1] = 1$
 $T[n] = T[n-1] + T[n-2]$ for $n \ge 2$

The characteristic polynomial is

$$\lambda^2-\lambda-1$$

Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2} \left(1 \pm \sqrt{5} \right)$$



Example: Fibonacci Sequence

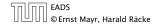
Hence, the solution is of the form

$$\alpha \left(\frac{1+\sqrt{5}}{2}\right)^n + \beta \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$T[0] = 0$$
 gives $\alpha + \beta = 0$.

T[1] = 1 gives

$$\alpha\left(\frac{1+\sqrt{5}}{2}\right)+\beta\left(\frac{1-\sqrt{5}}{2}\right)=1 \Longrightarrow \alpha-\beta=\frac{2}{\sqrt{5}}$$



Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$



Consider the recurrence relation:

 $c_0 T(n) + c_1 T(n-1) + c_2 T(n-2) + \dots + c_k T(n-k) = f(n)$ with $f(n) \neq 0$.

While we have a fairly general technique for solving homogeneous, linear recurrence relations the inhomogeneous case is different.

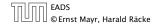


The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n)$$
 ,

where T_h is any solution to the homogeneous equation, and T_p is one particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.



Example:

$$T[n] = T[n-1] + 1$$
 $T[0] = 1$

Then,

$$T[n-1] = T[n-2] + 1 \qquad (n \ge 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n-1] = T[n-1] - T[n-2] \qquad (n \ge 2)$$

or

$$T[n] = 2T[n-1] - T[n-2] \qquad (n \ge 2)$$

I get a completely determined recurrence if I add T[0] = 1 and T[1] = 2.

Example: Characteristic polynomial:

$$\underbrace{\lambda^2 - 2\lambda + 1}_{(\lambda - 1)^2} = 0$$

Then the solution is of the form

$$T[n] = \alpha 1^n + \beta n 1^n = \alpha + \beta n$$

T[0] = 1 gives $\alpha = 1$.

$$T[1] = 2$$
 gives $1 + \beta = 2 \Longrightarrow \beta = 1$.

If f(n) is a polynomial of degree r this method can be applied r + 1 times to obtain a homogeneous equation:

$$T[n] = T[n-1] + n^2$$

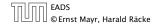
Shift:

$$T[n-1] = T[n-2] + (n-1)^2 = T[n-2] + n^2 - 2n + 1$$

Difference:

$$T[n] - T[n-1] = T[n-1] - T[n-2] + 2n - 1$$

$$T[n] = 2T[n-1] - T[n-2] + 2n - 1$$



$$T[n] = 2T[n-1] - T[n-2] + 2n - 1$$

Shift:

$$T[n-1] = 2T[n-2] - T[n-3] + 2(n-1) - 1$$
$$= 2T[n-2] - T[n-3] + 2n - 3$$

Difference:

$$T[n] - T[n-1] = 2T[n-1] - T[n-2] + 2n - 1$$
$$- 2T[n-2] + T[n-3] - 2n + 3$$

$$T[n] = 3T[n-1] - 3T[n-2] + T[n-3] + 2$$

and so on...