Technique 1: Round the LP solution.

We first solve the LP-relaxation and then we round the fractional values so that we obtain an integral solution.

Set Cover relaxation:

$$\begin{array}{lrcll}
\min & & \sum_{i=1}^{k} w_i x_i & & \\
\text{s.t.} & \forall u \in U & \sum_{i: u \in S_i} x_i & \geq & 1 \\
& \forall i \in \{1, \dots, k\} & x_i & \in & [0,1]
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Rounding Algorithm:

Set all x_i-values with $x_i \geq \frac{1}{f}$ to 1. Set all other x_i-values to 0.

Technique 1: Round the LP solution.

Lemma 2

The rounding algorithm gives an f-approximation.

Proof: Every $u \in U$ is covered.

- We know that $\sum_{i: u \in S_i} x_i \geq 1$.
- The sum contains at most $f_u \leq f$ elements.
- Therefore one of the sets that contain u must have $x_i \geq 1/f$.
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Technique 2: Rounding the Dual Solution.

Relaxation for Set Cover

Primal:

$$\begin{array}{lrl} \min & \sum_{i\in I} w_i x_i & \\ \text{s.t. } \forall u & \sum_{i:u\in S_i} x_i & \geq 1 \\ & x_i & \geq 0 \end{array}$$

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Technique 2: Rounding the Dual Solution.

Rounding Algorithm:

Let I denote the index set of sets for which the dual constraint is tight. This means for all $i \in I$

$$\sum_{u:u\in S_i} y_u = w_i$$

Technique 2: Rounding the Dual Solution.

Lemma 3

The resulting index set is an f-approximation.

Proof:

Every $u \in U$ is covered.

- Suppose there is a u that is not covered.
- This means $\sum_{u:u \in S_j} y_u < w_j$ for all sets S_j that contain u.
- But then y_u could be increased in the dual solution without violating any constraint. This is a contradiction to the fact that the dual solution is optimal.

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$$\begin{aligned}\sum_{i\in I} w_i &= \sum_{i\in I}\sum_{u:u\in S_i} y_u \\ &= \sum_u |\{i\in I : u\in S_i\}|\cdot y_u \\ &\le \sum_u f_u y_u\end{aligned}$$

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&\le f\,\mathrm{cost}(x^*) \\
&\le f \cdot \mathrm{OPT}
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Let I denote the solution obtained by the first rounding algorithm and I' be the solution returned by the second algorithm. Then

$$I \subseteq I' \ .$$

This means I' is never better than I.

- Suppose that we take S_i in the first algorithm. I.e., $i \in I$.
- This means $x_i^* \geq \frac{1}{f}$.
- Because of Complementary Slackness Conditions the corresponding constraint in the dual must be tight.
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Technique 3: The Primal Dual Method

The previous two rounding algorithms have the disadvantage that it is necessary to solve the LP. The following method also gives an f-approximation without solving the LP.

For estimating the cost of the solution we only required two properties.

1. The solution is dual feasible and, hence,

$$\sum_u y_u \leq \text{cost}(x^*) \leq \text{OPT}$$

   where x^* is an optimum solution to the primal LP.

2. The set I contains only sets for which the dual inequality is tight.

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Technique 3: The Primal Dual Method

Algorithm 1 PrimalDual

```
y ← 0
I ← ∅
while exists u ∉ ⋃_{i∈I} S_i do
    increase dual variable y_i until constraint for some
    new set S_ℓ becomes tight
    I ← I ∪ {ℓ}
```

Technique 4: The Greedy Algorithm

Algorithm 1 Greedy

$I \leftarrow \emptyset$
$\hat{S}_j \leftarrow S_j$ for all j
while I not a set cover **do**
  $\ell \leftarrow \arg\min_{j:\hat{S}_j \neq 0} \frac{w_j}{|\hat{S}_j|}$
  $I \leftarrow I \cup \{\ell\}$
  $\hat{S}_j \leftarrow \hat{S}_j - S_\ell$ for all j

In every round the Greedy algorithm takes the set that covers remaining elements in the most cost-effective way.

We choose a set such that the ratio between cost and still uncovered elements in the set is minimized.

Technique 4: The Greedy Algorithm

Lemma 4

Given positive numbers $a_1, \ldots, a_k$ *and* $b_1, \ldots, b_k$ *then*

$$\min_i \frac{a_i}{b_i} \le \frac{\sum_i a_i}{\sum_i b_i} \le \max_i \frac{a_i}{b_i}$$

Technique 4: The Greedy Algorithm

Let n_ℓ denote the number of elements that remain at the beginning of iteration ℓ. $n_1 = n = |U|$ and $n_{s+1} = 0$ if we need s iterations.

In the ℓ-th iteration

$$\min_j \frac{w_j}{|S_j|} \le \frac{\sum_{j\in \mathrm{OPT}} w_j}{\sum_{j\in \mathrm{OPT}} |S_j|} = \frac{\mathrm{OPT}}{\sum_{j\in \mathrm{OPT}} |S_j|} \le \frac{\mathrm{OPT}}{n_\ell}$$

since an optimal algorithm can cover the remaining n_ℓ elements with cost OPT.

Let $\hat{S}_j$ be a subset that minimizes this ratio. Hence, $w_j/|\hat{S}_j| \le \frac{\mathrm{OPT}}{n_\ell}$.

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Adding this set to our solution means $n_{\ell+1} = n_\ell - |\hat{S}_j|$.

$$w_j \leq \frac{|\hat{S}_j|\mathrm{OPT}}{n_\ell} = \frac{n_\ell - n_{\ell+1}}{n_\ell} \cdot \mathrm{OPT}$$

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&= H_n\cdot \mathrm{OPT} \le \mathrm{OPT}(\ln n+1)\ .
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Technique 5: Randomized Rounding

One round of randomized rounding:
Pick set S_j uniformly at random with probability $1 - x_j$ (for all j).

Version A: Repeat rounds until you have a cover.

Version B: Repeat for s rounds. If you have a cover STOP.
Otherwise, repeat the whole algorithm.

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$$\begin{aligned}\Pr[u \text{ not covered in one round}] &= \prod_{j:u\in S_j}(1-x_j) \le \prod_{j:u\in S_j} e^{-x_j} \\ &= e^{-\sum_{j:u\in S_j} x_j}\end{aligned}$$

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Probability that $u \in U$ is not covered (after ℓ rounds):

$$\Pr[u \text{ not covered after } \ell \text{ round}] \le \frac{1}{e^\ell} \enspace .$$

$\Pr[\exists u \in U \text{ not covered after } \ell \text{ round}]$

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$$= \Pr[u_1 \text{ not covered} \lor u_2 \text{ not covered} \lor \ldots \lor u_n \text{ not covered}]$$

$$
\begin{aligned}
&\Pr[\exists u \in U \text{ not covered after } \ell \text{ round}] \\
&\qquad = \Pr[u_1 \text{ not covered} \vee u_2 \text{ not covered} \vee \ldots \vee u_n \text{ not covered}] \\
&\qquad \le \sum_i \Pr[u_i \text{ not covered after } \ell \text{ rounds}]
\end{aligned}
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Lemma 5

With high probability $\mathcal{O}(\log n)$ rounds suffice.

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With high probability:

For any constant α the number of rounds is at most $\mathcal{O}(\log n)$ with probability at least $1 - n^{-\alpha}$.

Proof: We have

$$\Pr[\#\text{rounds} \geq (\alpha+1)\ln n] \leq ne^{-(\alpha+1)\ln n} = n^{-\alpha} \ .$$

Expected Cost

- Version A.
  Repeat for $s = (\alpha + 1) \ln n$ rounds. If you don't have a cover simply take all sets.

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If the weights are polynomially bounded (smallest weight is 1), sufficiently large α and OPT at least 1.

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E[\text{cost}] = \Pr[\text{success}] \cdot E[\text{cost} \mid \text{success}] \\
+ \Pr[\text{no success}] \cdot E[\text{cost} \mid \text{no success}]
\end{aligned}
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$$E[\text{cost} \mid \text{success}] = \frac{1}{\Pr[\text{sucess}]}\Big(E[\text{cost}] - \Pr[\text{no success}] \cdot E[\text{cost} \mid \text{no success}]\Big)$$

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&\le \frac{1}{\Pr[\text{sucess}]} E[\text{cost}] \le \frac{1}{1 - n^{-\alpha}} (\alpha + 1)\ln n \cdot \text{cost}(LP)
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for $n \geq 2$ and $\alpha \geq 1$.

Randomized rounding gives an $\mathcal{O}(\log n)$ approximation. The running time is polynomial with high probability.

Theorem 6 (without proof)

There is no approximation algorithm for set cover with approximation guarantee better than $\frac{1}{2}\log n$ unless NP has quasi-polynomial time algorithms (algorithms with running time $2^{\mathrm{poly}(\log n)}$).

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Techniques:

- Deterministic Rounding
- Rounding of the Dual
- Primal Dual
- Greedy
- Randomized Rounding
- Local Search
- Rounding the Data + Dynamic Programming