Degeneracy Revisited

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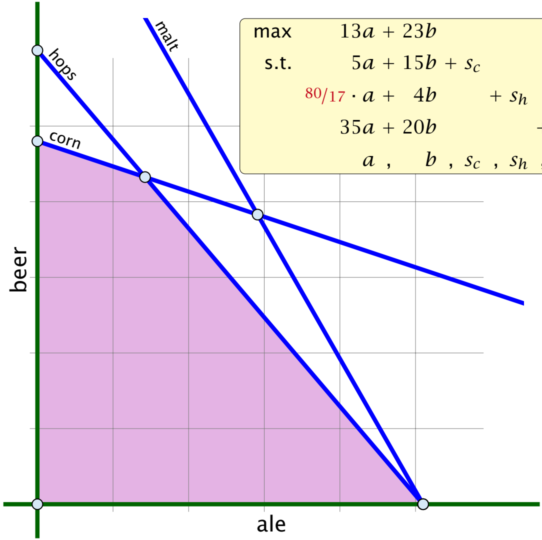
Degenerate Example



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\text{s.t.} & 5a & + 15b & + s_c & & & = & 480 \\
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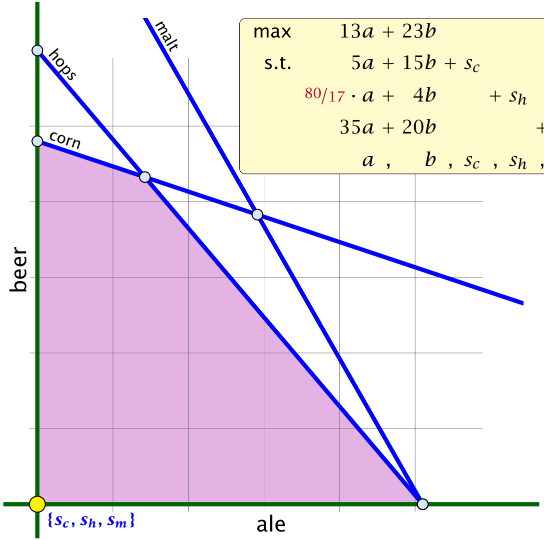

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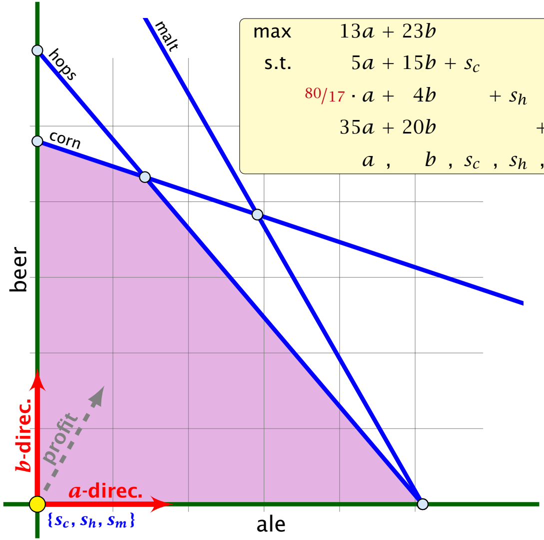

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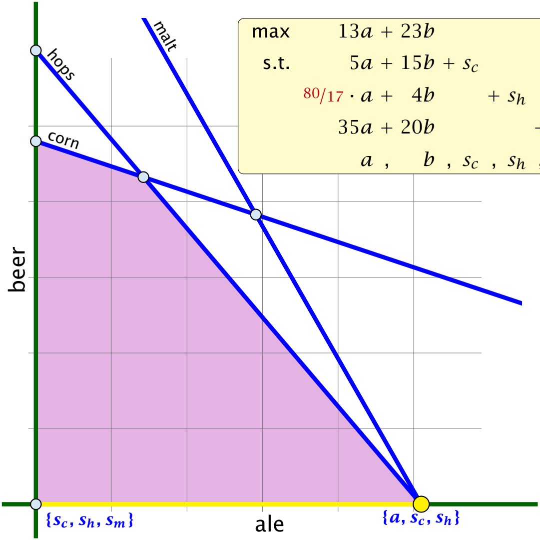

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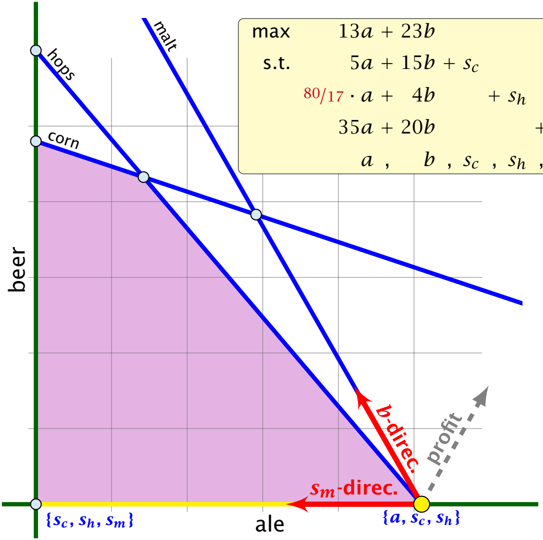

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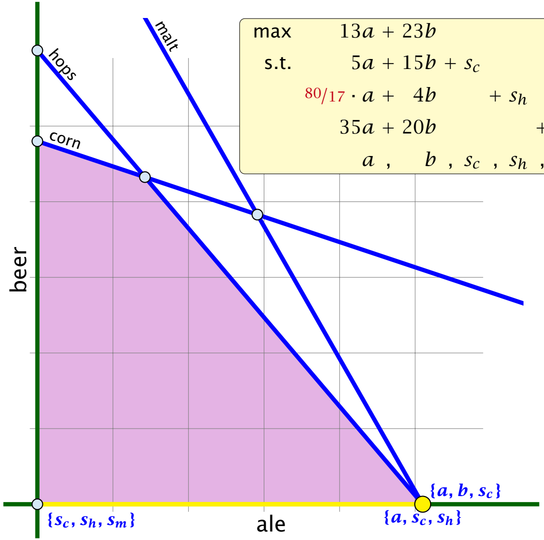

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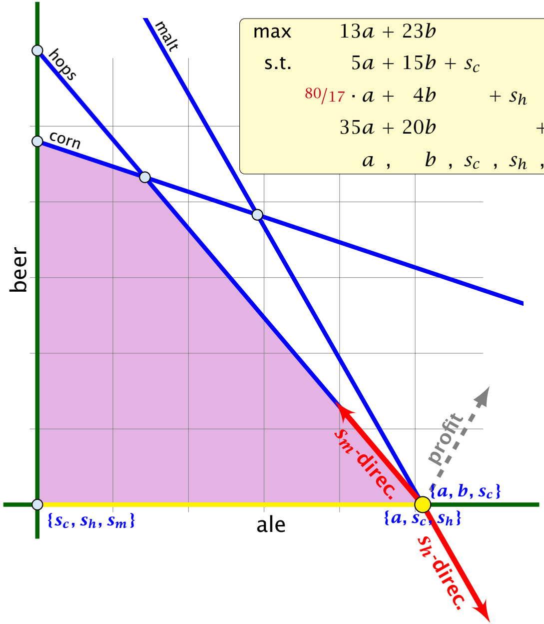

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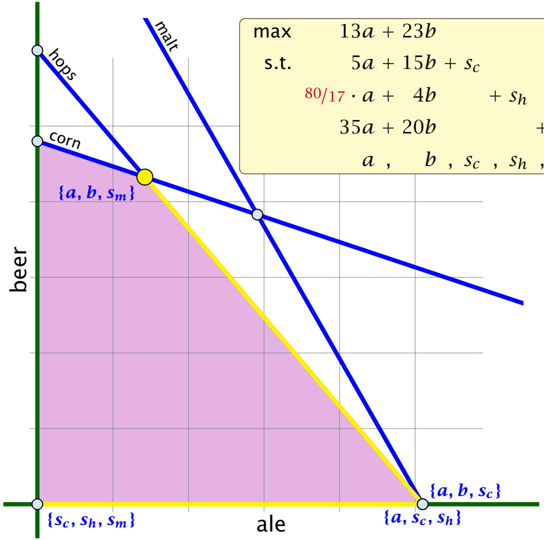

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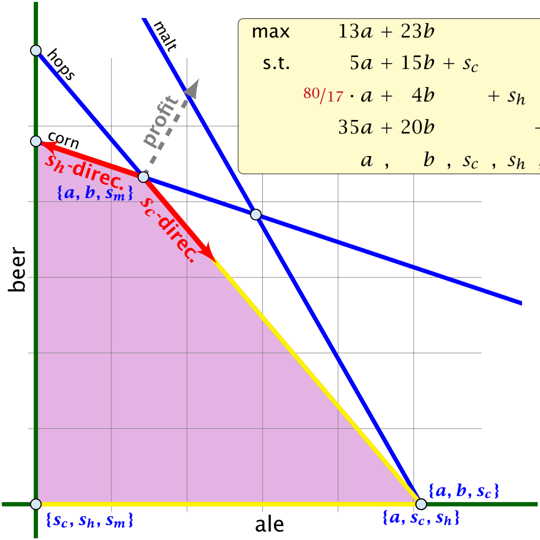

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- If a set B of basis variables corresponds to an infeasible basis (i.e. $A_B^{-1}b \not\geq 0$) then B corresponds to an infeasible basis in LP' (note that columns in A_B are linearly independent).
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Perturbation

Let B be index set of some basis with basic solution

$$x_B^* = A_B^{-1} b \geq 0, x_N^* = 0 \quad \text{(i.e. } B \text{ is feasible)}$$

Fix

$$b' := b + A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \text{ for } \varepsilon > 0 \ .$$

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The new LP is feasible because the set B of basis variables provides a feasible basis:

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Hence, $\tilde{B}$ is not feasible.

Property III

Let $\tilde{B}$ be a basis. It has an associated solution

$$x^*_{\tilde{B}} = A_{\tilde{B}}^{-1} b + A_{\tilde{B}}^{-1} A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}$$

in the perturbed instance.

We can view each component of the vector as a polynom with variable ε of degree at most m.

$A_{\tilde{B}}^{-1} A_B$ has rank m. Therefore no polynom is 0.

A polynom of degree at most m has at most m roots (Nullstellen).

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- If it terminates because it finds a variable x_j with $\tilde{c}_j > 0$ for which the j-th basis direction d, fulfills $d \geq 0$ we know that LP' is unbounded. The basis direction does not depend on b. Hence, we also know that LP is unbounded.

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Simulate behaviour of LP' without explicitly doing a perturbation.

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We choose the entering variable arbitrarily as before ($\tilde{c}_e > 0$, of course).

If we do not have a choice for the leaving variable then LP' and LP do the same (i.e., choose the same variable).

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In the following we assume that $b \geq 0$. This can be obtained by replacing the initial system $(A_B \mid b)$ by $(A_B^{-1}A \mid A_B^{-1}b)$ where B is the index set of a feasible basis (found e.g. by the first phase of the Two-phase algorithm).

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Matrix View

Let our linear program be

$$\begin{array}{rcrcl} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{rcrcl} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

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LP chooses an arbitrary leaving variable that has $\hat{A}_{\ell e} > 0$ and minimizes

$$\theta_\ell = \frac{\hat{b}_\ell}{\hat{A}_{\ell e}} = \frac{(A_B^{-1}b)_\ell}{(A_B^{-1}A_{*e})_\ell} \ .$$

ℓ is the index of a leaving variable within B. This means if e.g. $B = \{1, 3, 7, 14\}$ and leaving variable is 3 then $\ell = 2$.

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Definition 2
$u \leq_{\text{lex}} v$ if and only if the first component in which u and v differ fulfills $u_i \leq v_i$.

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$$= \frac{\ell\text{-th row of } A_B^{-1}(b \mid I)}{(A_B^{-1}A_{*e})_\ell}\begin{pmatrix}1\\ \varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix}$$

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This means you can choose the variable/row ℓ for which the vector

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Of course only including rows with $(A_B^{-1}A_{*e})_\ell > 0$.

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