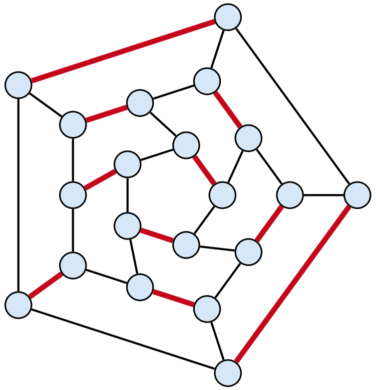
Part V

Matchings

Matching

- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Maximum Matching: find a matching of maximum cardinality



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16 Bipartite Matching via Flows

Which flow algorithm to use?

- Generic augmenting path: $\mathcal{O}(m \operatorname{val}(f^*)) = \mathcal{O}(mn)$.
- Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.
- Shortest augmenting path: $\mathcal{O}(mn^2)$.

For unit capacity simple graphs shortest augmenting path can be implemented in time $\mathcal{O}(m\sqrt{n})$.

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17 Augmenting Paths for Matchings

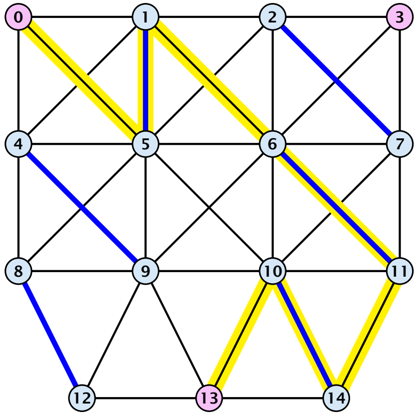
Definitions.

- Given a matching M in a graph G, a vertex that is not incident to any edge of M is called a free vertex w. r. .t. M.
- For a matching M a path P in G is called an alternating path if edges in M alternate with edges not in M.
- An alternating path is called an augmenting path for matching M if it ends at distinct free vertices.

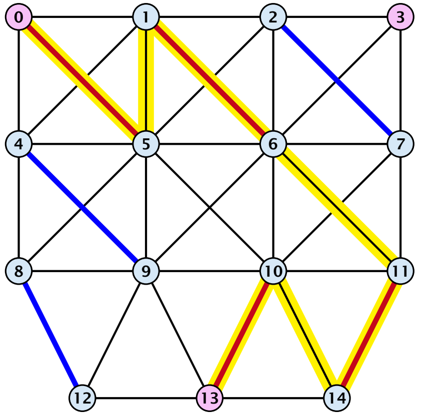
Theorem 1

A matching M is a maximum matching if and only if there is no augmenting path w. r. t. M.

Augmenting Paths in Action



Augmenting Paths in Action



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17 Augmenting Paths for Matchings

Proof.

⇒ If M is maximum there is no augmenting path P, because we could switch matching and non-matching edges along P. This gives matching $M' = M \oplus P$ with larger cardinality.

⇐ Suppose there is a matching M' with larger cardinality. Consider the graph H with edge-set $M' \oplus M$ (i.e., only edges that are in either M or M' but not in both).

Each vertex can be incident to at most two edges (one from M and one from M'). Hence, the connected components are alternating cycles or alternating path.

As $|M'| > |M|$ there is one connected component that is a path P for which both endpoints are incident to edges from M'. P is an alternating path.

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17 Augmenting Paths for Matchings

Algorithmic idea:

As long as you find an augmenting path augment your matching using this path. When you arrive at a matching for which no augmenting path exists you have a maximum matching.

Theorem 2

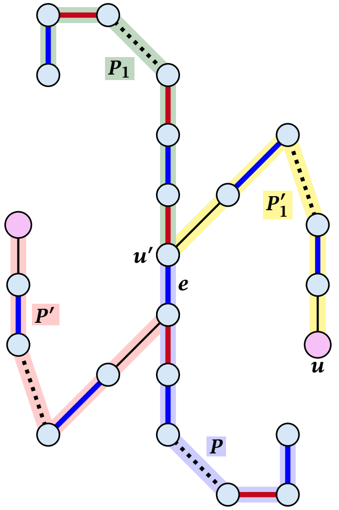
Let G be a graph, M a matching in G, and let u be a free vertex w.r.t. M. Further let P denote an augmenting path w.r.t. M and let $M' = M \oplus P$ denote the matching resulting from augmenting M with P. If there was no augmenting path starting at u in M then there is no augmenting path starting at u in M'.

The above theorem allows for an easier implementation of an augmenting path algorithm. Once we checked for augmenting paths starting from u we don't have to check for such paths in future rounds.

17 Augmenting Paths for Matchings

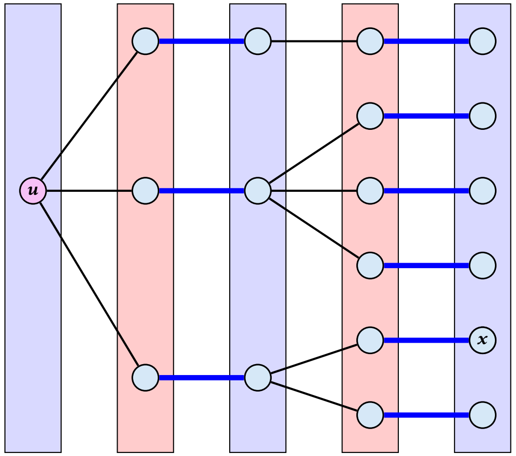
Proof

- Assume there is an augmenting path P' w.r.t. M' starting at u.
- If P' and P are node-disjoint, P' is also augmenting path w.r.t. M ($\lightning$).
- Let u' be the first node on P' that is in P, and let e be the matching edge from M' incident to u'.
- u' splits P into two parts one of which does not contain e. Call this part P_1. Denote the sub-path of P' from u to u' with P_1'.
- $P_1 \circ P_1'$ is augmenting path in M ($\lightning$).



How to find an augmenting path?

Construct an alternating tree.



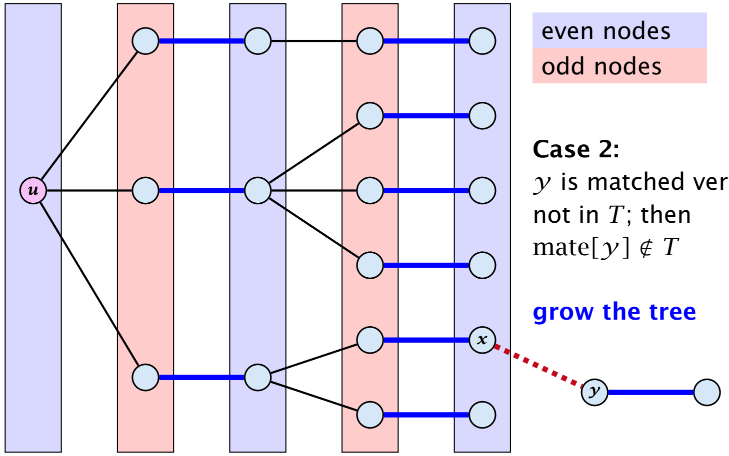
even nodes
odd nodes

Case 1:
y is free vertex not contained in T

you found alternating path

How to find an augmenting path?

Construct an alternating tree.



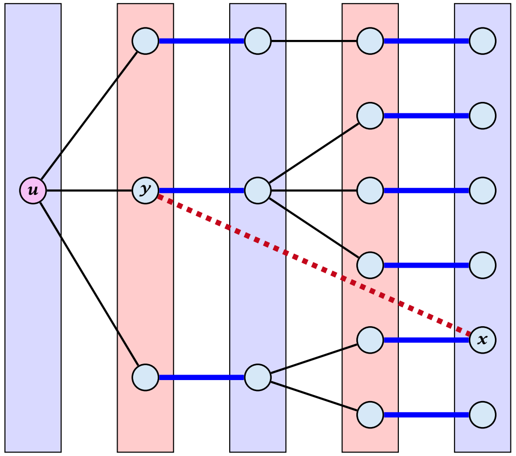
even nodes
odd nodes

Case 2:
y is matched vertex not in T; then $\text{mate}[y] \notin T$

grow the tree

How to find an augmenting path?

Construct an alternating tree.



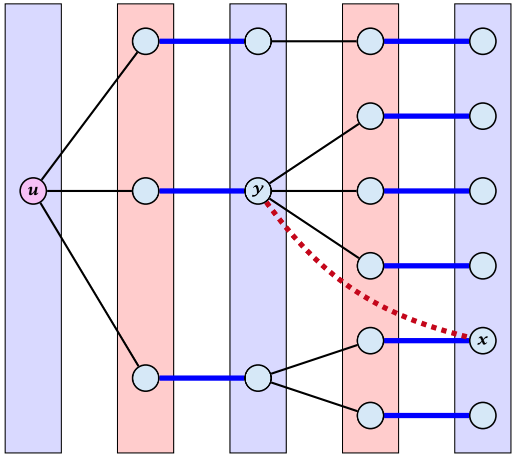
even nodes
odd nodes

Case 3:
y is already contained in T as an odd vertex

ignore successor y

How to find an augmenting path?

Construct an alternating tree.



even nodes
odd nodes

Case 4:
y is already contained in T as an even vertex

can't ignore y

does not happen in bipartite graphs

Algorithm 23 BiMatch$(G, match)$

```
for $x \in V$ do $mate[x] \leftarrow 0$;
$r \leftarrow 0$; $free \leftarrow n$;
while $free \geq 1$ and $r < n$ do
    $r \leftarrow r + 1$
    if $mate[r] = 0$ then
        for $i = 1$ to $n$ do $parent[i'] \leftarrow 0$
        $Q \leftarrow \emptyset$; $Q.\mathrm{append}(r)$; $aug \leftarrow \mathrm{false}$;
        while $aug = \mathrm{false}$ and $Q \neq \emptyset$ do
            $x \leftarrow Q.\mathrm{dequeue}()$;
            for $y \in A_x$ do
                if $mate[y] = 0$ then
                    augm$(mate, parent, y)$;
                    $aug \leftarrow \mathrm{true}$;
                    $free \leftarrow free - 1$;
                else
                    if $parent[y] = 0$ then
                        $parent[y] \leftarrow x$;
                        $Q.\mathrm{enqueue}(mate[y])$;
```

graph $G = (S \cup S', E)$

$S = \{1, \dots, n\}$

$S' = \{1', \dots, n'\}$

The lecture version of the slides contains a step-by-step explanation of the algorithm.

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18 Weighted Bipartite Matching

Weighted Bipartite Matching/Assignment

- Input: undirected, bipartite graph $G = L \cup R, E$.
- an edge $e = (\ell, r)$ has weight $w_e \geq 0$
- find a matching of maximum weight, where the weight of a matching is the sum of the weights of its edges

Simplifying Assumptions (wlog [why?]):

- assume that $|L| = |R| = n$
- assume that there is an edge between every pair of nodes $(\ell, r) \in V \times V$
- can assume goal is to construct maximum weight perfect matching

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Weighted Bipartite Matching

Theorem 3 (Halls Theorem)
A bipartite graph $G = (L \cup R, E)$ has a perfect matching if and only if for all sets $S \subseteq L$, $|\Gamma(S)| \geq |S|$, where $\Gamma(S)$ denotes the set of nodes in R that have a neighbour in S.