#### How to choose augmenting paths?

- We need to find paths efficiently.
- We want to guarantee a small number of iterations.

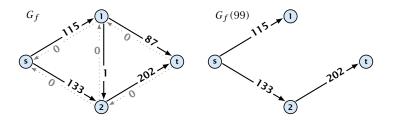
#### Several possibilities:

- Choose path with maximum bottleneck capacity.
- Choose path with sufficiently large bottleneck capacity.
- Choose the shortest augmenting path.



### Intuition:

- Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- Don't worry about finding the exact bottleneck.
- Maintain scaling parameter  $\Delta$ .
- $G_f(\Delta)$  is a sub-graph of the residual graph  $G_f$  that contains only edges with capacity at least  $\Delta$ .





11.3 Capacity Scaling

```
Algorithm 2 maxflow(G, s, t, c)
 1: foreach e \in E do f_e \leftarrow 0;
 2: \Delta \leftarrow 2^{\lceil \log_2 C \rceil}
 3: while \Delta \ge 1 do
 4: G_f(\Delta) \leftarrow \Delta-residual graph
5: while there is augmenting path P in G_f(\Delta) do

6: f \leftarrow \text{augment}(f, c, P)

7: \text{update}(G_f(\Delta))

8: \Delta \leftarrow \Delta/2
 9: return f
```



#### Assumption:

All capacities are integers between 1 and C.

#### Invariant:

All flows and capacities are/remain integral throughout the algorithm.

### Correctness:

The algorithm computes a maxflow:

- because of integrality we have  $G_f(1) = G_f$
- therefore after the last phase there are no augmenting paths anymore
- this means we have a maximum flow.



**Lemma 1** *There are*  $\lceil \log C \rceil$  *iterations over*  $\Delta$ *.* **Proof:** obvious.

### Lemma 2

Let f be the flow at the end of a  $\Delta$ -phase. Then the maximum flow is smaller than  $val(f) + m\Delta$ .

Proof: less obvious, but simple:

- There must exist an *s*-*t* cut in  $G_f(\Delta)$  of zero capacity.
- In  $G_f$  this cut can have capacity at most  $m\Delta$ .
- This gives me an upper bound on the flow that I can still add.



### Lemma 3

There are at most 2m augmentations per scaling-phase.

Proof:

- Let *f* be the flow at the end of the previous phase.
- $\operatorname{val}(f^*) \leq \operatorname{val}(f) + 2m\Delta$
- Each augmentation increases flow by  $\Delta$ .

Theorem 4

We need  $\mathcal{O}(m \log C)$  augmentations. The algorithm can be implemented in time  $\mathcal{O}(m^2 \log C)$ .

