# **Overview: Shortest Augmenting Paths**

#### Lemma 1

The length of the shortest augmenting path never decreases.

### Lemma 2

After at most  $\mathcal{O}(m)$  augmentations, the length of the shortest augmenting path strictly increases.

### **Overview: Shortest Augmenting Paths**

These two lemmas give the following theorem:

### Theorem 3

The shortest augmenting path algorithm performs at most  $\mathcal{O}(mn)$  augmentations. This gives a running time of  $\mathcal{O}(m^2n)$ .

### Proof.

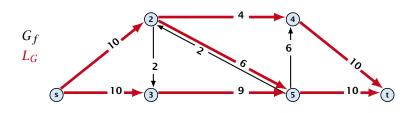
- We can find the shortest augmenting paths in time  $\mathcal{O}(m)$  via BFS.
- O(m) augmentations for paths of exactly k < n edges.



Define the level  $\ell(v)$  of a node as the length of the shortest s-v path in  $G_f$ .

Let  $L_G$  denote the subgraph of the residual graph  $G_f$  that contains only those edges (u, v) with  $\ell(v) = \ell(u) + 1$ .

A path P is a shortest s-u path in  $G_f$  if it is a an s-u path in  $L_G$ .



In the following we assume that the residual graph  $\mathcal{G}_f$  does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

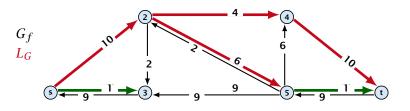
#### First Lemma:

The length of the shortest augmenting path never decreases.

After an augmentation  $G_f$  changes as follows:

- Bottleneck edges on the chosen path are deleted.
- Back edges are added to all edges that don't have back edges so far.

These changes cannot decrease the distance between s and t.

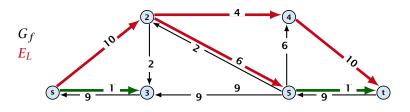


**Second Lemma:** After at most m augmentations the length of the shortest augmenting path strictly increases.

Let  $E_L$  denote the set of edges in graph  $L_G$  at the beginning of a round when the distance between s and t is k.

An s-t path in  $G_f$  that uses edges not in  $E_L$  has length larger than k, even when considering edges added to  $G_f$  during the round.

In each augmentation one edge is deleted from  $E_L$ .



### **Theorem 4**

The shortest augmenting path algorithm performs at most  $\mathcal{O}(mn)$  augmentations. Each augmentation can be performed in time  $\mathcal{O}(m)$ .

### Theorem 5 (without proof)

There exist networks with  $m = \Theta(n^2)$  that require O(mn) augmentations, when we restrict ourselves to only augment along shortest augmenting paths.

#### Note:

There always exists a set of m augmentations that gives a maximum flow (why?).

When sticking to shortest augmenting paths we cannot improve (asymptotically) on the number of augmentations.

However, we can improve the running time to  $\mathcal{O}(mn^2)$  by improving the running time for finding an augmenting path (currently we assume  $\mathcal{O}(m)$  per augmentation for this).

We maintain a subset  $E_L$  of the edges of  $G_f$  with the guarantee that a shortest s-t path using only edges from  $E_L$  is a shortest augmenting path.

With each augmentation some edges are deleted from  $E_L$ .

When  $E_L$  does not contain an s-t path anymore the distance between s and t strictly increases.

Note that  $E_L$  is not the set of edges of the level graph but a subset of level-graph edges.

Suppose that the initial distance between s and t in  $G_f$  is k.

 $E_L$  is initialized as the level graph  $L_G$ .

Perform a DFS search to find a path from s to t using edges from  $E_L$ .

Either you find t after at most n steps, or you end at a node v that does not have any outgoing edges.

You can delete incoming edges of v from  $E_L$ .

Let a phase of the algorithm be defined by the time between two augmentations during which the distance between  $\emph{s}$  and  $\emph{t}$  strictly increases.

Initializing  $E_L$  for the phase takes time  $\mathcal{O}(m)$ .

The total cost for searching for augmenting paths during a phase is at most  $\mathcal{O}(mn)$ , since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in  $E_L$  and takes time  $\mathcal{O}(n)$ .

The total cost for performing an augmentation during a phase is only  $\mathcal{O}(n)$ . For every edge in the augmenting path one has to update the residual graph  $G_f$  and has to check whether the edge is still in  $E_L$  for the next search.

There are at most n phases. Hence, total cost is  $\mathcal{O}(mn^2)$ .