Fundamental Algorithms

Chapter 8: Graphs

Harald Räcke

Winter 2015/16

Graphs

Definition (Graph)

A **graph** $G = (V, E)$ consists of a set V of vertices (nodes) and a set E of edges between the vertices.

- **undirected graph**: $(i, j) \in E$ an unordered pair – $(i, j) = (j, i)$
- **directed graph** (or shorter: "digraph"):
  $(i, j) \in E$ an ordered tuple, i.e. $(i, j) \in E$ independent of $(j, i) \in E$

Graphs

Definition (Graph)

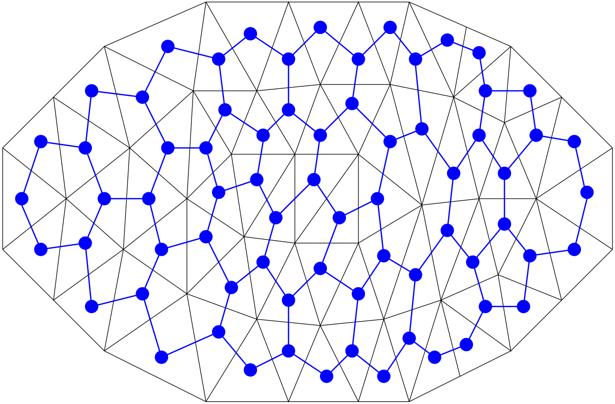
A **graph** $G = (V, E)$ consists of a set V of vertices (nodes) and a set E of edges between the vertices.

- **undirected graph**: $(i, j) \in E$ an unordered pair – $(i, j) = (j, i)$
- **directed graph** (or shorter: "digraph"): $(i, j) \in E$ an ordered tuple, i.e. $(i, j) \in E$ independent of $(j, i) \in E$

Some Terms

- two vertices V_0 and V_n are connected by a **path** (of length n), if there is a sequence of edges $(V_0, V_1), (V_1, V_2), \ldots, (V_{n-1}, V_n)$
- a graph is **connected**, if there is a path between any two vertices
- a vertex V has **degree** d, if V has d (outgoing) edges

Graphs in CSE – Unstructured Grids:



- in blue: V = grid cells, E = neighbours (“dual graph”)
- in black: V = grid vertices, E = cell edges

Trees

Definition (Tree)

A **tree** is a connected graph without cycles.

Trees

Definition (Tree)

A **tree** is a connected graph without cycles.

$\rightarrow$ *Question: is this consistent with our "naive" image of a tree?*

Trees

Definition (Tree)

A **tree** is a connected graph without cycles.

$\rightarrow$ *Question: is this consistent with our "naive" image of a tree?*

Theorem

A graph T is a ***tree****, if and only if there is a unique path between any two distinct vertices of T.*

Trees

Definition (Tree)

A **tree** is a connected graph without cycles.

$\rightarrow$ *Question: is this consistent with our "naive" image of a tree?*

Theorem

A graph T is a **tree***, if and only if there is a unique path between any two distinct vertices of T.*

Implications:

- there is only one connection from the root to any of the nodes
- any path between two nodes will run through the root of the resp. subtree
- actually: which node is the "root" ?

Trees (2)

Theorem

A connected graph (V, E) *is a tree, if and only if* $|E| = |V| - 1$

Trees (2)

Theorem

A connected graph (V, E) *is a tree, if and only if* $|E| = |V| - 1$

Implications:

- if you “cut” one edge, a tree is no longer connected (child becomes an orphan)
- building a tree incrementally requires a root (one node, no edge) and one additional edge per added node

Trees (2)

Theorem

A connected graph (V, E) is a tree, if and only if $|E| = |V| - 1$

Implications:

- if you “cut” one edge, a tree is no longer connected (child becomes an orphan)
- building a tree incrementally requires a root (one node, no edge) and one additional edge per added node

Definition (Spanning Tree)

$T = (V, E)$ is called a **spanning tree** for the graph $G = (V, E')$, if T is a tree, and $E \subset E'$.

Note: T has the same vertices as G.

Data Structures for Graphs

Pointer-Based Data Structure: (esp. for directed graphs)

```
Node := (
    key: Integer,
    edges: List of Node );
}
```

Data Structures for Graphs

Pointer-Based Data Structure: (esp. for directed graphs)

```
Node := (
    key: Integer,
    edges: List of Node );
}
```

Adjacency Matrix:

- $n \times n$ matrix A, where $n = |V|$
- $a_{ij} = 1$, if $(i, j) \in E$
- A symmetric for undirected graphs

Data Structures for Graphs

Pointer-Based Data Structure: (esp. for directed graphs)

```
Node := (
    key: Integer,
    edges: List of Node );
}
```

Adjacency Matrix:

- $n \times n$ matrix A, where $n = |V|$
- $a_{ij} = 1$, if $(i, j) \in E$
- A symmetric for undirected graphs

Note: to store an adjacency matrix as an $n \times n$ array is a good idea,
only if $|E| \in \Theta(n^2)$

Graph Traversals

Definition (Graph Traversal:)

Input: a (connected!) directed or undirected graph (V, E), and a node $x \in V$.
Task: Starting from x, "visit" all vertices in V (following edges only)

Graph Traversals

Definition (Graph Traversal:)

Input: a (connected!) directed or undirected graph (V, E), and a node $x \in V$.
Task: Starting from x, "visit" all vertices in V (following edges only)

Examples:

- modify the key values of all vertices
- search a specific key value in a graph

Graph Traversals

Definition (Graph Traversal:)

Input: a (connected!) directed or undirected graph (V, E), and a node $x \in V$.
Task: Starting from x, “visit” all vertices in V (following edges only)

Examples:

- modify the key values of all vertices
- search a specific key value in a graph

Two main variants:

- depth-first traversal (depth-first search)
- breadth-first traversal (breadth-first search)

Depth-First Traversal

```
DFTraversal(V:Node) {
   ! mark current node V as visited:
   Mark[V.key] = 1;
   ! perform desired work on V:
   Visit(V);
   ! perform traversal from all nodes connected to V
   forall (V,W) in V.edges do
      if Mark[W.key] = 0 then DFTraversal(W);
   end do;
}
```

Assumptions:

- keys $V.key$ numbered from $1, \dots, n = |V|$
- Mark : **Array**[1..n]
- **forall** loop executed sequentially

DF-Traversal – Stack-Based Implementation

```
StackDFTrav(X:Node) {
   ! uses stack of "active" nodes
   Stack active = { X }; Mark[X.key] = 1;
   while active <> {} do
      ! remove first node from stack
      V = pop(active);
      Visit(V);
      forall (V,W) in V.edges do
         if Mark[W] = 0 then {
           push(active, W); Mark[W.key] = 1;
         }
      end do;
   end while;
}
```

$\rightarrow$ use stack as last-in-first-out (LIFO) data container

Breadth-First-Traversal

Queue-Based Implementation

```
BFTraversal(X:Node) {
   ! uses queue of "active" nodes
   Queue active = { X }; Mark[X.key] = 1;
   while active <> {} do
      ! remove first node from queue
      V = remove(active);
      Visit(V);
      forall (V,W) in V.edges do
         if Mark[W.key] = 0 then {
            append(active, W); Mark[W.key] = 1;
         }
      end do;
   end while;
}
```

→ use queue as first-in-first-out (FIFO) data container

DF-Traversal – Stack-Based Implementation

```
StackDFTrav(X:Node) {
   ! uses stack of "active" nodes
   Stack active = { X };
   while active <> {} do
      ! remove first node from stack
      V = pop(active);
      if Mark[V.key] = 0 then {
         Visit(V); Mark[V.key] = 1;
         forall (V,W) in V.edges do
            push(active, W);
         end do;
      }
   end while;
}
```

→ use stack as last-in-first-out (LIFO) data container

Breadth-First-Traversal

Queue-Based Implementation

```
BFTraversal(X:Node) {
   ! uses queue of "active" nodes
   Queue active = { X };
   while active <> {} do
      ! remove first node from queue
      V = remove(active);
      if Mark[V.key] = 0 then {
          Visit(V); Mark[V.key] = 1;
          forall (V,W) in V.edges do
             append(active, W);
          end do;
      }
   end while;
}
```

→ use queue as first-in-first-out (FIFO) data container

Breadth-First Search

```
BFSearch(x:Node, k:Integer) : Node {
   Queue active = { x };
   while active <> {} do
      V = remove(active);
      if V.key = k then return V;
      if Mark[V.key] = 0 then
         Mark[V.key] = 1
         forall (V,W) in V.edges do
            append(active, W);
         end do;
      end if;
   end while;
}
```

Breadth-First Search

```
BFSearch(x:Node, k:Integer) : Node {
  Queue active = { x };
  while active <> {} do
    V = remove(active);
    if V.key = k then return V;
    if Mark[V.key] = 0 then
      Mark[V.key] = 1
      forall (V,W) in V.edges do
        append(active, W);
      end do;
    end if;
  end while;
}
```

Breadth-First Search as Shortest-Path Algorithm:

- breadth-first search will return the node with the **shortest path** from x
- requires modification of algorithm to return this path, as well

Breadth-First and Depth-First Traversal

DF/BF-Traversal and Connectivity of Graphs:

- DF- and BF-traversal will visit all nodes of a connected graph
- if a non-connected graph is traversed, both algorithms can be used to find the (maximum) connected sub-graph that contains the start node
- hence, DF- and BF-traversal can be extended to find all connectivity components of a graph

Breadth-First and Depth-First Traversal

DF/BF-Traversal and Connectivity of Graphs:

- DF- and BF-traversal will visit all nodes of a connected graph
- if a non-connected graph is traversed, both algorithms can be used to find the (maximum) connected sub-graph that contains the start node
- hence, DF- and BF-traversal can be extended to find all connectivity components of a graph

DF/BF-Traversal and Trees:

- DF- and BF-traversal will compute a spanning tree of a connected graph
- BF-traversal generates a spanning tree with shortest paths